A JOURNEY THROUGH THE LOGIC WONDERLAND

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A Journey through the Logic Wonderland

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September, 2016

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This is the outcome of a real-time travel through the beautiful landscape of logic revealed in the morphology of informatics. A group of researchers of Department of Information Technology of IIEST, Shibpur and their teachers took a journey along with one Professor of logic who had experienced this walk several times before. But the togetherness, unhindered exchanges and spirit of the wanderers gave rise to something that exceeded most anticipations. Especially with the assembly of young hearts down the line, the acquired intensity and resonance of shades of colour in the joy of creation increased manifold. And the notation of that symphony is presented here.

Foreword

When Dr. Sukanta Das approached me to deliver some lectures on logic to his Ph.D students and himself I readily agreed. This was because first, I love talking on logic and second, the audience consisted of matured minds who do research in a different (but not totally detached) field. I wanted to undergo a process different from what I had been doing for long years in my standard logic classes in mathematics departments. So, my only condition to Sukanta and his students was that they should interact. And they did so for which I very much enjoyed the classes. Evidences of interaction may be marked in repetition, non-linear arrangement of the topics, incompleteness (claims without justification since not all our discussions could be noted), lack in rigour, mis(ill)construction of sentences, different languages at different places (because notes were prepared by different persons), even some mistakes. When the manuscript was handed over to me I prepared to make changes as little as possible. After all, this was not intended to be a book, it was an account of our wonderful academic journey together. This is an outcome of our joint deliberation. To novices, delight of this journey might perhaps be shared. To experts even, the strange arrangement of the topics and passage from one topic to another might reveal some natural flow of queries that they usually do not encounter or can not entertain. My thanks to Sukanta, his students, Subhasis (Banerjee) and Mallika (Sarbadhikari) for offering me this opportunity.

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Preface

It was a hot summer in Kolkata when we met Professor Mihir K Chakraborty in a small class room of IT department for the first time. The room was not so hot; but the irritative weather was pinching all of us. And Professor Chakraborty started to talk on Logic! We didn't imagine at the first meet that what an enchanting journey we had already started! And the journey lasted for more than one year. This lecture note is a rough account of that exciting journey.

We were six regular participants in this lecture series – me and five of my PhD students: Kamalika, Raju, Souvik, Sumit and Supreeti. Sometime Professor Subhasis Bandopadhyay of HSS department of this Institute, Dr. Biswanath Sethi, my former student and a faculty member of Indira Gandhi Institute of Technology, Sarang, Odisha, and Dr. Subhasis Chakraborty, a physician by profession joined us in the lecture. We six are the learners and artisans of computing science, and work particularly in the domain of cellular automata. So we were very much interested to learn Mathematical Logic along with its philosophical aspects from a mathematician who is not only a true Master in this domain but also a pioneer researcher of the area. Some topics of Mathematical Logic are generally covered in the courses of Computer Science and Information Technology. So we also wanted to re-look at the subject through the eyes of a mathematician. Professor Mihir Kumar Chakraborty (MKC) is not a mathematician only; he is a philosopher, and a dreamer. And we got a reflection of this side of his persona throughout the journey. When MKC started to talk, we immediately realized that it was not going to be a traditional class on Logic. He frequently referred to the philosophical debates and brought various philosophical ideas in his lectures. He insisted us to make the journey interactive and collaborative so that we could feel that we all were players in the game! There was no known flow in the lecture – a spontaneity overwhelmed the journey which touched different sectors of Logic in its own rhythm. We interacted; we debated, and got motivation to cope up with new ideas and research challenges. Yes, the journey was a joint venture, but the leadership of a beautiful mathematician enabled us to discover so many shades of Logic!

Let us take a brief look at the journey. At the beginning, it was appealed to the audience that what they intuitively understand by "Logic". From this intuitive understanding, what we can expect from Mathematics of Logic was discussed. We were introduced first the semantic definition of Logic. Material implication, symbolized by $\longrightarrow$, is a point of debate in Philosophical Logic since long. The debate was cherished by us and decided that we would arrange a survey to know what people in general think about implication. And, we got strange results! Even the professors didn't agree to the traditional understanding of implication! However, we next moved to the syntactic definition of logic. And these two definitions led us to soundness and completeness theorems of Logic. We started with Propositional Logic, then visited to the

First Order Logic. In the midway, we took a quick look at the Modal Propositional Logic. We would come back to the Modal Logic again at the end where System K, System T etc. would be discussed. However, the most exciting part of the journey might be the visits to Number Theory and Gödel's Incompleteness Theorem. Many stories and mysteries about Gödel were nurtured in the class. Few years back, MKC wrote a book in Bengali on Gödel, from where he shared his views on Gödel's works. It is, however, always exciting to re-look at the mathematical theories through the eyes of First Order Logic and Set Theory. MKC argued that First Order Logic is sufficient in mathematics, and there is no need to go for higher order Logic!

After that, the journey moved beyond the classical (Two-valued) logic. At its first appearance, the Three-Valued Propositional Logic was discussed. And then we peeked into the multi-valued logic. This new areas of Logic, also known as Non Standard Logic which is the domain of MKC's research, had frequently come in his lectures. For example, Paraconsistent Logic, Sequent Calculus etc. had been referred in the journey as outcomes of limitations of classical logics. Finally, we visited the Fuzzy Logic.

The lecture series was started on the mid of July of 2016, and continued until the March of 2018. After the start of the journey, however, we decided to develop a lecture note which will be shared among the interested people. My five students divided the job among them, prepared the notes which were then shown to MKC for his approval. He always wanted to keep our understanding intact in the writings, keeping aside his own views. We do not know how much justice we were able to do to this incredible journey, but we tried to give a feel of this (non standard) journey to the readers. We placed the topics in this note as it happened in the class room. And this has made it different from any other standard books or class notes on Logic.

We are very thankful to Department of Humanities and Social Sciences (HSS) for sponsoring Professor Mihir Kumar Chakraborty as Visiting Professor during that period. We announced in Indian School of Logic and Applications ISLA (Part II), held in this Institute on December, 2018, that we would publish this class note on Web, which is finally going to happen. I personally am delighted. I am thankful to my students who prepared this notes with patience. I have no words to express my gratitude to Professor Chakraborty, and I shall surely not try to do that. I consider myself as a happy student of him, and want to further continue this journey with him. If anybody gets any benefit from this lecture, we shall consider that our effort has been successful. I am sure that Sir will also feel the same.

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Contents

Chapter 1

Introduction to Logic

Class 1: Dated 15 **- July -** 2016

What do we understand by logic? An intuitive answer may be, “that follows an inbuilt reason and comes to a conclusion”. However, in general, any subject deals with two basic philosophical questions: *What?* and *How?*. The domain of knowledge that deals with the question “what?” is called *Ontology* and the same for the question “how?” is called *Epistemology*. So, in the above question, we are mainly concerned about ontology of Logic.

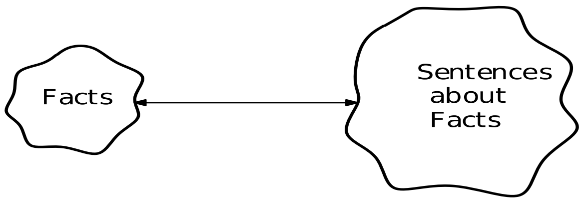


Figure 1.1: Relation between Facts and Sentences

- True/ False: This can be said of the sentences about facts. e.g. - ‘Snow is white’ is true. Similarly, ‘4 is prime’ is false.

Right now, we will consider only the type of logic that admits values either true or false. So, comes the question which entities have the value true/false? The facts or the sentences? Actually, ‘facts’ and ‘the sentences about facts’ are two different sets (see Fig.1.1). True/False can be ascribed only to the linguistic description of the facts, because these are associated with sentences about facts, not facts themselves. If a sentence properly describes the fact, the value is ‘True’. True/False are properties of sentences.

Theory of Truth/ Philosophy of Truth/Correspondence Theory of Truth *[A. Tarski]*

The sentence $\underbrace{\text{“}\textit{Snow is white}\text{''}}_{\textit{Sentence}}$ *is true if and only if* $\underbrace{\textit{snow is white}}_{\textit{Fact}}$.

Sentences are linguistic descriptions belonging to some language(s) and relate to some fact(s). But, how can we assign a truth value to a sentence? It can be done by *verification.* For example, for the sentence “Snow is white”, one can verify the color of snow as white. Take another sentence “ 4 is prime”; here, the verification can be done by computation. Note that, the method of verification is not mentioned in the correspondence theory of truth. It is the question ‘How’, that is, part of Epistemology.

Above two examples are *atomic* or *rudimentary* sentences. However, determination of truth/falsehood of such atomic sentences is not a part/task of logic. Logic deals with sentence(s) made of more than one atomic sentence, e.g. *not p*, *p or q*, *p and q* ,*if p then q*.

Sentences are to mean a fact, which is connected to an Ontology. However, there can be two other outcomes of sentences with respect to truth:

- Neither: This happens, for example, for *future contingent statements*, first mentioned by Aristotle. Note that, ‘neither’ is the case, when we see it in terms of Epistemology. In terms of Ontology, there is either true/false. If one can foresee, she can write the value. An example is “Tomorrow there shall be rain”. This statement can be neither true nor false now, as it is going to happen tomorrow, i.e., can't be determined in the present time.
- Both: The value of a statement can be both true and false, e.g.- ‘A’ is a terrorist. This gets the truth value depending on the circumstances / analogies /context.

Now, two questions readily come out - (i) Are facts related to atomic sentences? (ii) What is the fact related with a complex non-atomic sentence? These can be answered by an example.

Example 1 “Man is mortal”

Here, the fact is every man is mortal. But, “man” is not an entity, it is a set or collective that we build for our purpose. In the given sentence, we are creating facts for our own *purposes* to draw a *conclusion*, which is, all men are mortal. Note that, the property ‘mortal’ is contained in ‘Man’, however, set theoretically, the thing is opposite. The set ‘man’ is contained in the set ‘mortal’.

1.1 Distributive and Non-distributive Plurals

In some sentences, a property applied over the plural form of an entity can be applied to its singular form also keeping the same meaning, that is, the relationship is the same for both the singular and plural forms. Such forms are termed as *distributive plurals*. However, in some sentences, this is not true; and the form is called *non-distributive plural*. The name "distributive" comes from the fact that, here the form predicate distributes over individual entities.

Example 2 "Men are mortal" *if and only if* "Every man is mortal". Here, the meaning is implied. So, it is an example of distributive plural.

However, the sentence "Students offered a bouquet to their teacher", does not imply that each individual student has offered a bouquet. So, it is non-distributive.

Set theory is non-distributive. What is associated with a set, may not be associated with each of its elements.

Example 3 Take two sets $\{2, 3, 4\}$ and $\{1, 2, 3, 4\}$. Here, $\{2, 3, 4\} \subseteq \{1, 2, 3, 4\}$. However, $2 \nsubseteq \{1, 2, 3, 4\}$, $3 \nsubseteq \{1, 2, 3, 4\}$ and $4 \nsubseteq \{1, 2, 3, 4\}$. So, "is a subset of" is non-distributive to the elements of the set.

1.2 Connectives

$\wedge$ (AND), $\vee$ (OR), $\neg$ (NOT), $\rightarrow$ (IF $\cdots$ THEN) are called logical connectives that give rise to complex sentences out of simpler sentences. Here X is a set of sentences and α is one sentence. The sign $\vdash$ denotes a relationship between them, termed as "*follow(s)*". For example, let X be a set of the following two sentences:

1. 2 is prime.
2. If 2 is prime, then 3 is odd.

and α is the sentence "4 is even". Now, the problem is whether α *follows* from X or not. That is, whether "4 is even" follows from the two sentences, namely '2 is prime' and 'If 2 is prime, then 3 is odd' of X. Incidentally, all the three sentences are true, yet we shall see that, the sentence α does not *logically* follow from the sentences 1 and 2.

Task for the reader: Find statistics about public view on whether α follows from these two sentences.

How is the relationship ‘Follows’ understood in Logic? Actually, the purpose of logic is to *precisify* (understand) the natural language word ‘*Follow*’. We can intuitively understand the meaning of ‘Follow’. But, how can we define (precisify) the term ‘Follow’ in logic?

In logic, the set of sentences, X is called the *premises* and α is the *conclusion*. The relationship *follow(s)* is termed as consequence and is denoted by $\vdash$ (*turnstile*) (see Fig. 1.2).

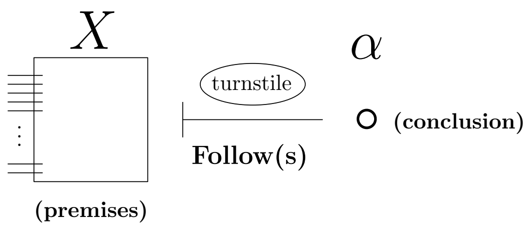


Figure 1.2: Premise Conclusion Relation

Note that, in the above example, while 1 and α are atomic sentences, 2 is non-atomic.

1.2.1 Definition of turnstile ($\vdash$)

Definition 1 *$X \vdash \alpha$ holds if and only if, for all situations, whenever all the members of X are ‘true’, α is also ‘true’.*

X is called the premise and α is called the conclusion.

If we know the truth values of the (atomic) sentences, then the truth values of the premise and the conclusion can be derived. One mode of obtaining the truth values is by using a *truth-table*. It is the specification or rule given by logic. For example, if the truth values of $p \wedge q$, $p \vee q$, $\neg p$ and $p \rightarrow q$ are to be found, we can construct the following truth-table of Table 1.1.

However, that set of statements X entails a statement α, does not mean that each statement in the premise entails α. For example, in Fig. 1.3, if in any situation, the statements of X are true, then for that situation, the conclusion q is also true. Thus the conclusion q ‘follows’ from X. However, ‘X entails α’ does not mean each member of X entails α. This is similar to the non-distributive plurals. The rule of logic, shown in Fig. 1.3, is called *modus ponens*.

In the following example of Fig. 1.4, an extra premise r is added to X, which

Table 1.1: Example of Logic using Truth-table

		p and q	**p or q**	**not p**	**if p then q**
p	q	$p \wedge q$	$p \vee q$	$\neg p$	$p \to q$
True	True	True	True	False	True
True	False	False	True	False	False
False	True	False	True	True	True
False	False	False	False	True	True

$$\overset{X}{\boxed{\begin{array}{l} p \\ p \to q \end{array}}} \quad \vdash \quad q \qquad \textbf{[modus ponens]}$$

Figure 1.3: Modus ponens

is not used for arriving at the conclusion. However, for the relationship 'follows' to hold, we need to consider the situations where r is true as well as p and $p \to q$ are true'; that is, the situations with 'r false' and '$p \;\; \& \;\; p \to q$ true' will not be counted. Note that, increase in premise means smaller class of situations to hold; 'r false' and '$p \;\; \& \;\; p \to q$ true' will not be a situation to consider.

$$\overset{X}{\boxed{\begin{array}{l} r \\ p \\ p \to q \end{array}}} \quad \vdash \quad q$$

Figure 1.4: Another example of logic

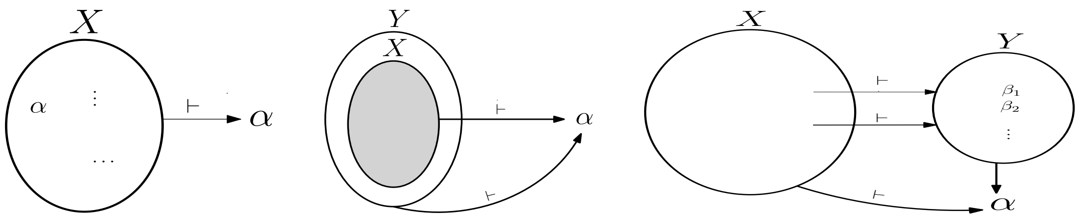
1.2.2 Properties of turnstile ($\vdash$)

For a $\vdash$ *(turnstile)* to be defined, the following three properties must be satisfied.

1. If $\alpha \in X$ then $X \vdash \alpha$. [Reflexivity/Overlap]
2. If $X \vdash \alpha$ then $Y \vdash \alpha$, when $X \subseteq Y$. [Monotonicity/Dilution]
3. If $X \vdash \beta \in Y$, $forall\ \beta$, and $Y \vdash \alpha$, then $X \vdash \alpha$. [Cut/General transitivity]

One can see that $\vdash$ defined as in Def 1 satisfies the above three conditions.

The following figures Fig. 1.5a, Fig. 1.5b and Fig. 1.5c depict the situations for the three properties respectively. Note that, the example of Fig. 1.4 holds because of monotonicity/dilution property.



(a) Reflexivity or overlap property of turnstile($\vdash$)

(b) Monotonicity or dilution property of turnstile($\vdash$)

(c) Cut or general transitivity property of turnstile($\vdash$)

Figure 1.5: Properties of turnstile($\vdash$)

Now, we can formally define logic in the following section.

1.3 Logic

To define logic, we need sentences. Let us consider $\mathcal{L}$ to be the set of all possible sentences. This $\mathcal{L}$ is called the language. Language can be natural, such as English, Bengali etc., or formal. For the purpose of Logic, we use formal language. We shall formally define *formal* language in next chapter.

Logic can be formally defined by a pair $(\mathcal{L}, \vdash)$. So, logic is a language and turnstile relation, where $\vdash$ satisfies properties (1), (2) and (3). In a logic, the left hand side of the relation $\vdash$ represents a set of sentences of the language and right hand side represents a sentence of the language.

Here, $X \subseteq \mathcal{L}$, $\alpha \in \mathcal{L}$ and $\vdash$ is a consequence relation.

Mathematically, turnstile ($\vdash$) is a binary relation from power set of $\mathcal{L}$ to $\mathcal{L}$. That is, $\vdash$: $\mathcal{P}(\mathcal{L})$) to $\mathcal{L}$. Anyway, if we consider the same $\mathcal{L}$ and change the $\vdash$ relation, we get a new logic. Now the question is, how many logics can we get for a given $\mathcal{L}$?

Since $\mathcal{L}$ is a set of all possible sentences, $\mathcal{L}$ is taken to be an infinite set. But this set is countable. However, $\mathcal{P}(\mathcal{L})$ is an uncountable set. Hence, the number of possible turnstile relations is uncountably many. So, we can get uncountably many logics against a given $\mathcal{L}$.

1.4 Some points to be noted

- The purpose or task of logic is to study the turnstile/follow(s) relation, not finding the truth value of the (atomic) sentences.
- In case of logical OR ($\vee$) operation, there is a situation where both the operands are true. But, this situation usually does not come in practical cases. For example, the answer to the question "Tea or coffee" is either 'Tea' or 'Coffee', so either one is 'true', not both. Such situations are examples of *Exclusive OR* operation.
- There are different modes of computing truth values.
- *Object* and *Property* are two disjoint notions.
- An atomic Fact is an object with a property.
- *Plurals* is a new and on-going domain of research.
- Mathematics for distributive plurals is still an open issue.

Chapter 2

Propositional Logic: Semantic Definition

Class 2: Dated 22 **- July -** 2016

According to Chapter 1 logic has been defined by a pair $(\mathcal{L}, \vdash)$, where $\mathcal{L}$ is the Language and $\vdash$ is the consequence relation or the inference engine. '$\vdash$' follows Overlap, Monotonicity and Cut properties.

Notion of truth:
Notion of truth shows how to know whether $X \vdash \alpha$ holds or not. For that, the correspondence theory is also required.

Correspondence theory:
Whether a statement is true or false is decided by correspondence with reality. For example, "Snow is white" is true if and only if snow is white. (Tarski)

However, we need to define a language in more concrete terms.

2.1 Formal Language

Formal language can be defined informally as

- Basic set of symbols (i.e. Alphabet).
- Sentences by arranging alphabet or we can say, strings of symbols taken from the alphabet.

Example 4 As an example, "I go home". Now consider the gaps in the sentence. There are gaps between characters as well as words, but here we shall consider the gaps between words and use a symbol for the gaps. Let the symbol be $\#$. Therefore, the sentence becomes "I$\#$go$\#$home": string of symbols (with gap $\#$) taken from the alphabet. We shall keep a count for the length of the string.

The length of the string "I#go#home" is 9, because we have 9 occurrences of symbols involved in it. To make such sentences, some rules are to be followed which are collectively said to be "grammar".

Definition 2 *A formal language is a set of string of symbols taken from alphabet, A i.e. a language $\mathcal{L} \subseteq A^*$, A^* is a set of all strings over A. In order to form a language, it is not essential for A to be finite.*

Example 5 As an example, see Figure 2.1

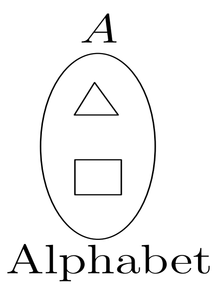


Figure 2.1: Set of alphabet

Formation rules:
Now, we can set the rules for the language $\mathcal{L}$ where $A = \{\triangle, \square\}$.

i) Any string is finite.

ii) A string is acceptable if and only if it starts with $\triangle$ and ends with two successive $\square$'s.

Example 6 Then, we can say $\triangle\ \square\square$ is accepted ($\triangle\ \square\square \in \mathcal{L}$) and $\triangle\ \square\ \triangle$ is not accepted ($\triangle\ \square\ \triangle \notin \mathcal{L}$).

Once the alphabet is specified and rules are specified, then it is decidable whether a string is a part of language or not. For technical reasons we shall also consider empty string as a string which will be denoted by Λ. Empty string is not 'blank' or '#'.

2.2 Propositional logic

Definition 3 $(\mathcal{L}, \vdash)$ *where $\mathcal{L}$ is the language. The alphabet $\mathcal{A}$ of $\mathcal{L}$ is given by, $A = \{p, |, \neg, \rightarrow, (,)\}$, where the formation rules to arrange the alphabet are:*

1. *Any p followed by a finite number of $|$ marks is in $\mathcal{L}$.*

2. *If α is in $\mathcal{L}$ then $(\neg\alpha)$ is in $\mathcal{L}$.*

3. *If α, β are in $\mathcal{L}$ then $(\alpha \to \beta)$ is in $\mathcal{L}$.*

4. *Nothing else is in $\mathcal{L}$.*

Note that such a definition is known as recursive definition.

Example 7 According to rule (1), $p|$, $p||$, $p|||$, $\cdots$, these are atomic well formed formula (wffs). These represent actual simple sentences, like - 'I go home', '2 is prime' etc.

A well formed formula (wff) is a finite sequence of symbols from a given alphabet which may or may not be part of a formal language. In our case (Definition 3), alphabet is not a part of language

Example 8 According to, rule (2), $(\neg p|)$, $(\neg p||)$, $(\neg(\neg p|))$, $\cdots$, these are well formed formula (wffs).

Note that, the meaning of

- $(\neg\alpha)$: *negation of* α
- $(\alpha \to \beta)$: *if* α *then* β

Now, we can make the correspondence after having the interpretation of the symbols with natural language. Hence, all the atomic sentences, via interpretation get truth values either 'T' (True) or 'F' (False).

$\alpha \to \beta$ is a new sentence composed of two sentences α and β, not a relationship. So, $\alpha \to \beta$ should not be read as "α *implies* β".

Example 9 As an example,

- $p|$: snow is white (T)
- $p|$: 3 is even (F),
  where T/F is decided by 'Correspondence' theory of truth
- if $p|$ is F, then according to the truth table (Table 1.1, page 5) $(\neg p|)$ is T (So, role of logic comes after getting truth values of wff.).

Suppose, $p|$ is F and $p||$ also F, then $p| \to p||$ (gives $F \to F$). Now the question: is this a true implication?

Valuation function:

Let v be a mapping that maps every atomic sentence to T or F, that is it maps all the atomic sentence to a 2-valued set $\{T, F\}$. Usually, for a specific topic, the number of atomic sentence are finite.

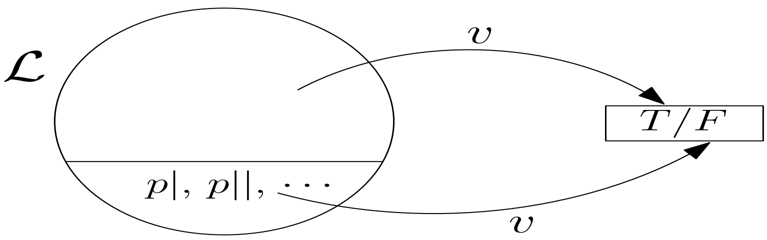


Figure 2.2: Mapping of every atomic sentence to T or F

This function is extended over the whole set $\mathcal{L}$.

Example 10 Using the truth tables (Table 1.1) we can get examples as below-

1. $p| \to \neg p||$ and $v(p|) = T$, $v(p||) = T$.
   Then, $v(p| \to \neg p||) = v(p|) \to \neg v(p||) = F$

2. $(p|| \to p|) \to (p| \to \neg p||)$ and $v(p|) = F$, $v(p||) = T$.
   Then, $v((p|| \to p|) \to (p| \to \neg p||))$
   $= v(p|| \to p|) \to v(p| \to \neg p||)$
   $= (v(p||) \to v(p|)) \to (v(p|) \to \neg v(p||))$
   $= (T \to F) \to (F \to F)$
   $= F \to T$
   $= T$

Definition 4 *If $\alpha, \beta \in \mathcal{L}$, then we write*

- $\alpha \wedge \beta$ *for* $\neg(\alpha \to \neg\beta)$
- $\alpha \vee \beta$ *for* $(\neg\alpha) \to \beta$

Definition 5 *$X \vdash \alpha$ holds if and only if for all valuation v, $v(x) = T$ for all $x \in X$ implies $v(\alpha) = T$. This means, if premise is true, then conclusion is true.*

In the next chapter, we shall discuss the notion elaborately.

Chapter 3

Propositional Logic: Semantic and Syntactic definition

Class 3: Dated 3 **- August -** 2016

The definition of logic is already given in the Chapters 1, 2. Turnstile '$\vdash$' is the relation which is defined by $X \vdash \alpha$ where $\alpha \in \mathcal{L}$, $X \subseteq \mathcal{L}$. We can say, α is a consequence of X. So, '$\vdash$' is called the consequence relation where X is said to be **premise** and α is said to be the **conclusion**.

3.1 Propositional Logic: Valuation function

Let '$\vdash$' be the consequence relation of classical logic with valuation v which is 2-valued. Then the valuation maps the atomic sentences to either 0 (False) or 1 (True) i.e. v: $A \rightarrow \{T, F\}$ where A is the set of all atomic formula. The extension of the valuation over the set of all wffs can be written as,

Definition 6

$$v(\neg\alpha) = \begin{cases} 1 & \textit{iff } v(\alpha) = 0 \\ 0 & \textit{iff } v(\alpha) = 1 \end{cases}$$

Definition 7

$$v(\alpha \rightarrow \beta) = \begin{cases} 0 & \textit{iff } v(\alpha) = 1 \textit{ and } v(\beta) = 0 \\ 1 & \textit{Otherwise} \end{cases}$$

Symbols 1 and 0 are often used for T(true) and F (false). Properties of the numbers 1 and 0 sometimes help in determining values of complex formulae.

Here, two basic symbols are taken: $\neg$ and $\rightarrow$ (Definition 6, 7). We shall introduce two more symbols- $\wedge$ (conjunction) and $\vee$ (disjunction). Based on the two symbols ($\neg$, $\rightarrow$) we define the following.

Definition 8 $\alpha \wedge \beta \equiv \neg(\alpha \rightarrow \neg\beta)$

Definition 9 $\alpha \vee \beta \equiv (\neg\alpha) \rightarrow \beta$

From definition 6 and 7 follow the equations given below:

$$v(\alpha \wedge \beta) = \begin{cases} 1 & \text{iff } v(\alpha) = 1 = v(\beta) \\ 0 & \text{Otherwise} \end{cases} \tag{3.1}$$

and

$$v(\alpha \vee \beta) = \begin{cases} 1 & \text{iff either } v(\alpha) \ \textit{or} \ v(\beta) = 1 \\ 0 & \text{iff } v(\alpha) = 0 = v(\beta) \end{cases} \tag{3.2}$$

Proof of equation 3.1: $\alpha \wedge \beta \equiv \neg(\alpha \rightarrow \neg\beta)$. Therefore, $v(\alpha \wedge \beta) = v(\neg(\alpha \rightarrow \neg\beta))$. $v(\neg(\alpha \rightarrow \neg\beta)) = 1$ if and only if $v(\alpha \rightarrow \neg\beta) = 0$. From 7, we know that, $v(\alpha \rightarrow \neg\beta) = 0$ iff $v(\alpha) = 1$ and $v(\neg\beta) = 0$ i.e. $v(\alpha) = 1$ and $v(\beta) = 1$. Hence proved.

Proof of equation 3.2: $\alpha \vee \beta \equiv (\neg\alpha) \rightarrow \beta$. Therefore, $v(\alpha \wedge \beta) = v((\neg\alpha) \rightarrow \beta)$. $v((\neg\alpha) \rightarrow \beta) = 0$ if and only if $v(\neg\alpha) = 1$ and $v(\beta) = 0$ (From Definition 7). From definition 6, we know that, $v(\neg\alpha) = 1$ if and only if $v(\alpha) = 0$. Therefore, $v((\neg\alpha) \rightarrow \beta) = 0$ if and only if $v(\alpha) = 0$ and $v(\beta) = 0$. Hence proved.

It is interesting to note that the truth tables for $\neg$, $\rightarrow$, $\wedge$, $\vee$ may now be expressed by operations on numerals 0 and 1 as $v(\neg\alpha) = 1 - v(\alpha)$, $v(\alpha \rightarrow \beta) = \max(1 - v(\alpha), v(\beta))$, $v(\alpha \wedge \beta) = \min(v(\alpha), v(\beta))$ and $v(\alpha \vee \beta) = \max(v(\alpha), v(\beta))$ respectively. The mystery of truth is captured by the mystery of numbers, Strange!

3.2 Tautology, Contradiction and Contingents

Definition 10 ***Tautology:*** *A tautology is a formula* α *that is true for every possible valuation. This means* $v(\alpha) = 1$ *for all* v.

Example 11 $\alpha \rightarrow (\beta \rightarrow \alpha)$

Example 12 $\alpha \vee (\neg\alpha)$

The above two formulae are true for any value of α and β, $v(\alpha), v(\beta) \in \{0, 1\}$.

The truth table of example 11 is given in table 3.1. It shows that the value of the formula in the example is always 1 and hence the formula, $\alpha \to (\beta \to \alpha)$, is a tautology.

Table 3.1: Truthtable of $\alpha \to (\beta \to \alpha)$

α	β	$\alpha \to (\beta \to \alpha)$
1	1	1
1	0	1
0	1	1
0	0	1

Definition 11 ***Contradiction:*** *A contradiction is a formula α such that $v(\alpha) = 0$ for all v.*

Example 13 $\neg(\alpha \to (\beta \to \alpha))$.

Table 3.2: Truthtable of $\neg(\alpha \to (\beta \to \alpha))$

α	β	$\neg(\alpha \to (\beta \to \alpha))$
1	1	0
1	0	0
0	1	0
0	0	0

A truth table of example 13 is given in table 3.2.

Definition 12 ***Contingents:*** *There are other well formed formulae (wffs) which are sometimes true and sometimes false. These wffs are known as Contingents.*

Example 14 $p| \wedge (\neg p||)$ is a contingent.

Table 3.3 represents a contingent for example 14.

Def. 5 of chapter 2 now reduces to a more precise mathematical statement viz. Def. 13.

Table 3.3: Truthtable of $p| \wedge (\neg p||)$

$p\vert$	$p\vert\vert$	$p\vert \wedge (\neg p\vert\vert)$
1	1	0
1	0	1
0	1	0
0	0	0

Definition 13 *$X \vdash \alpha$ holds if and only if for all valuation v, whenever $v(x) = 1$ for all $x \in X$, $v(\alpha) = 1$.*

It might be that for some premise $X \subseteq L$, there is no valuation v for which every member of X is true. In such a case, $X \vdash \alpha$ holds for all α.

Now, we can prove the Overlap, Dilution, Cut properties by using the valuation function. Here, we show the proof of Dilusion property.

Dilusion Property: If $X \vdash \alpha$ and $X \subseteq Y$, then $Y \vdash \alpha$ holds.

Proof: Let v be an arbitrary valuation and $v(Y) = 1$. Since, $X \subseteq Y$, so v (X) = 1. As $X \vdash \alpha$ is given, therefore $v(\alpha) = 1$ and hence $Y \vdash \alpha$ holds.

Exercise 1: Prove that the relation '$\vdash$' (relation with valuation) satisfies overlap and cut.

Say, $X \vdash \alpha$ is defined as above. Note that there has been no restriction on X. Naturally, question may arise what is meant by $\phi \vdash \alpha$ when $X = \varnothing$ (null set). It means that if $v(x) = 1$ for all $x \in \varnothing$, then $v(\alpha) = 1$. For any v, $v(x) = 1$ for all $x \in \varnothing$ is false as $\varnothing$ does not contain any value. So, for all v, $v(\alpha) = 1$ and hence α is a tautology.

Now, here come some questions. How many such tautologies one can have? Is it possible to describe everything by knowing only a few tautologies? How many of the tautologies are fundamental and how many of them can be described? It would be easy if some of the tautologies are identified and the rest could be written in terms of these tautologies because there exist actually infinitely many tautologies. But which one can be called basic tautologies among all the tautologies? Can we say some tautologies as basic? The answer is 'Yes'. We can say some tautologies as basic. If we can say that some tautologies are basic, it means we can group those basic tautologies. The basic tautologies can be extracted from all the tautologies and the other tautologies are derived from the basics. For this, $\vdash$ is used and this is the reason why axiomatic system is formed. Axiomatic system defines $\vdash$ syntactically. When I am defining the consequence semantically, I am an outsider. But when the definition is given syntactically, then I am an insider of the system.

Solve some questions given below.

Exercise 2: Find X, α such that $X \vdash \alpha$ does not hold.
Exercise 3: Is $\vdash$ countable or uncountable?

3.3 Defining '$\vdash$' without valuation: Axiomatic way of definition

Let A_x be a non-empty subset of wffs given by,

- Axiom 1: $\alpha \to (\beta \to \alpha)$
- Axiom 2: $(\alpha \to (\beta \to \gamma)) \to ((\alpha \to \beta) \to (\alpha \to \gamma))$
- Axiom 3: $(\neg\alpha \to \neg\beta) \to (\beta \to \alpha)$

where α, β, γ are any wffs. Also let us take a rule viz.

- Rule: set $\{\alpha, \alpha \to \beta\}$ is related to β (Definition of Modus Ponens).

All the three axioms and the rule mentioned above are syntactic that is dependent only on their forms. Based on these axioms and rule, we can now define the $\vdash_{A_x}$ relation.

Definition 14 *$X \vdash_{A_x} \alpha$ holds iff there is a sequence α_1, α_2, $\cdots$ α_n of wffs such that*

- $\alpha_n \equiv \alpha$
- *any α_i is either in*
  - A_x,
  - *or in X,*
  - *or is obtained by the rule i.e Modus Ponen (M.P) from the previous wffs of the sequence.*

Definition 15 ***Theorem/Thesis:*** *When $X = \phi$ (null set), i.e. $\phi \vdash_{A_x} \alpha$, then α is called the theorem or the thesis and it is represented by $\vdash_{A_x} \alpha$.*

Example 15 Establish $\alpha \to \alpha$ is a thesis i.e. $\vdash_{A_x} (\alpha \to \alpha)$

By definition 14, we need some wffs where the last wff is $\alpha \to \alpha$ and other wffs are either axioms or obtained by modus ponens from the previous wffs. The following sequence of wffs satisfies the conditions of the definition of thesis.

1. $\alpha \to ((\alpha \to \alpha) \to \alpha)$ (Axiom 1)
2. $(\alpha \to ((\alpha \to \alpha) \to \alpha)) \to ((\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha))$ (Axiom 2)
3. $(\alpha \to (\alpha \to \alpha)) \to (\alpha \to \alpha)$ (M.P on steps 1, 2)
4. $\alpha \to (\alpha \to \alpha)$ (Axiom 1)
5. $\alpha \to \alpha$ (M.P on steps 3, 4)

Hence, $\alpha \to \alpha$ is a thesis.

Example 16 Establish $\{\alpha, \alpha \to \beta, \beta \to \gamma\} \vdash \gamma$

Some wffs are given as follows:

1. α (in the Premise)
2. $\alpha \to \beta$ (in the Premise)
3. β (M.P on steps 1, 2)
4. $\beta \to \gamma$ (in the Premise)
5. γ (M.P on steps 3, 4)

The above sequence of wffs satisfies the conditions given in definition 14. Hence $\{\alpha, \alpha \to \beta, \beta \to \gamma\} \vdash \gamma$ holds.

Proposition: Given any axiom α, $\vdash_{A_x} \alpha$ holds i.e. α is a thesis.

Note that, all the axioms are theses, but all theses are not axioms. Figure 3.1 depicts that the set of axioms is a subset of the set of theses or theorems.

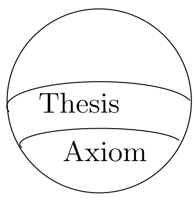


Figure 3.1: all axioms are theorem, but all theorem are not axioms

Now, we can prove that axiomatic relation $\vdash_{A_x}$ satisfies Overlap, Dilution, Cut properties. Here, we show that axiomatic relation '$\vdash_{A_x}$' satisfies 'cut'.

Theorem 1 *The axiomatic relation '$\vdash_{A_x}$' satisfies 'cut'.*

Recall: Cut, if $X \vdash y \in Y$ and $Y \vdash \alpha$, then $X \vdash \alpha$. Fig. 3.2 depicts the proof of Cut.

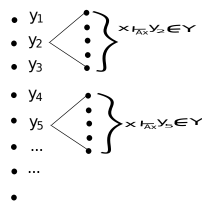


Figure 3.2: The chain of wffs satisfying the properties of $Y \vdash_A$x α transformed into a chain of wffs to show that $X \vdash_{A_x} \alpha$

Definition 16 ***Derivation of*** α ***from the Premise:*** *For $X \vdash_{A_x} \alpha$, if we get such a chain $\alpha_1, \alpha_2, \cdots, \alpha_n (\equiv \alpha)$, then the chain is called a derivation of α from the Premise X.*

Here everything is syntactic and it depends on the form only. It is useful for computer. The derivation of a formula from a set is not unique.

If $\vdash_{A_x} \equiv \vdash$, then we can say $X \vdash_{A_x} \alpha$ if and only if $X \vdash \alpha$. A machine will check for $X \vdash_{A_x} \alpha$ if it wants to verify whether $X \vdash \alpha$ is true or not. If $\phi \vdash_{A_x} \alpha$, then $\phi \vdash \alpha$ is a tautology.

- If $X \vdash_{A_x} \alpha$ is true then $X \vdash \alpha$ is also true. This is called **Soundness**.
- If $X \vdash \alpha$ is true then $X \vdash_{A_x} \alpha$ is also true. This is called **Completeness**.

This axiomatic system is called Hilbert type axiomatic system.

3.4 Exercise

- Prove that the relation '$\vdash_{A_x}$' (Axiomatic relation) satisfies overlap and dilusion.
- Is the relation '$\vdash_{A_x}$' is uncountably many? State the reason.

Chapter 4

Propositional Logic: Deduction Theorem

Class 4: Dated 9 **- August -** 2016

4.1 Deduction Theorem

Theorem 2 ***(Deduction Theorem)*** *If* Γ *is a set of wffs and* α *and* β *are wffs, and* $\Gamma \cup \{\alpha\} \vdash \beta$*, then* $\Gamma \vdash \alpha \to \beta$ *(Herbrand, 1930).*

Proof:

The proof is by induction on the length n of the derivation of β from $\Gamma \cup \{\alpha\}$. Let such a derivation be

$$\boxed{\begin{array}{c} \alpha_1 \\ \alpha_2 \\ . \\ . \\ \alpha_i \\ . \\ . \\ \alpha_n = \beta \end{array}}$$

Therefore, the wffs $\alpha_1, \alpha_2, \cdots, \alpha_i$ are either from Axioms or Premises or obtained by MP.

Now, to prove this theorem, we need *strong induction. By using weak induction we can not prove it.*

Strong induction and weak induction:

In weak induction we use "if p(k) is true then p(k+1) is true" while in strong induction we use "if p(i) is true for all i less than or equal to k then p(k+1) is true", where p(k) is some statement about the positive integer k.

Sketch of proof:

Theorem holds for the least element.
if the theorem holds for all n $\leqslant$ m then it holds for m+1
$\therefore$ Theorem holds for all n $\geqslant$ least element.

Basis step: n=1, So, three possibilities:

1: β is an Axiom.

2: β is Γ.

3: β is $\{\alpha\}$.

Case 1:

1. β (Axiom)
2. $\beta \to (\alpha \to \beta)$ (Axiom 1)
3. $\alpha \to \beta$ (MP)

So, The theorem holds.

Case 2:

1. β (Premises)
2. $\beta \to (\alpha \to \beta)$ (Axiom 1)
3. $\alpha \to \beta$ (MP)

So, The theorem holds.

Case 3: So, we need to derive $\alpha \to \alpha$ from Γ. Now, $\vdash_{\mathcal{A}_x} \alpha \to \alpha$ (already proved). So, $\alpha \to \alpha$. Therefore, the theorem holds.

If the theorem is true for all $n \leqslant m$ then it is true for $n = m + 1$.
Induction hypothesis: Assume that the theorem is true for all $n \leqslant m$
So, we shall show that, the theorem holds for n = m+1.

So, β may be derived from $\Gamma \cup \{\alpha\}$ through following steps.

1: β is an Axiom.

2: β is in Γ.

3: β is in $\{\alpha\}$.

4: β is obtained by MP.

- cases 1, 2, 3 as before.
- **case 4:**

$$Let, \Gamma \cup \{\alpha\} \vdash_{A_x} \alpha_i, i \leqslant m. \tag{4.1}$$

$$\Gamma \cup \{\alpha\} \vdash_{A_x} \alpha_j, j \leqslant m. \tag{4.2}$$

$$\boxed{\begin{array}{c} \alpha_1 \\ \alpha_2 \\ .. \\ \alpha_i = \alpha_i \\ ... \\ \alpha_j = \alpha_i \rightarrow \beta \\ \alpha_{m+1} = \beta \end{array}}$$

$$So, \Gamma \vdash_{A_x} \alpha \rightarrow \alpha_i$$

$$\Gamma \vdash_{A_x} \alpha \rightarrow \alpha_j$$

$$\Gamma \vdash_{A_x} \alpha \rightarrow (\alpha_i \rightarrow \beta)$$

Therefore,

1. $\alpha \rightarrow (\alpha_i \rightarrow \beta)$
2. $(\alpha \rightarrow (\alpha_i \rightarrow \beta)) \rightarrow ((\alpha \rightarrow \alpha_i) \rightarrow (\alpha \rightarrow \beta))$ [Axiom 2]
3. $(\alpha \rightarrow \alpha_i) \rightarrow (\alpha \rightarrow \beta)$ $[MP]$
4. $\alpha \rightarrow \alpha_i$

5. $\alpha \to \beta$ $[MP]$ (proved).

Deduction Theorem (Special case:) Let assume that $\Gamma = \varnothing$. Thus the statement is, if $\alpha \vdash_{A_x} \beta$ then $\varnothing \vdash_{\mathcal{A}_x} \alpha \to \beta$.

Theorem 3 ***(Converse of Deduction Theorem)*** *If Γ is a set of wffs and α and β are wffs, and $\Gamma \vdash_{A_x} \alpha \to \beta$ then $\Gamma \cup \{\alpha\} \vdash_{A_x} \beta$.*

Converse of Deduction Theorem (Special case:) If $\vdash_{\mathcal{A}_x} \alpha \to \beta$ then $\alpha \vdash_{A_x} \beta$.

4.2 Explosiveness property

We have to prove that,

$\vdash \neg\, \alpha \to (\alpha \to \beta)$.

Proof:

1. $(\neg\, \beta \to \neg\, \alpha) \to (\alpha \to \beta)$ [Axiom 3]
2. $((\neg\, \beta \to \neg\, \alpha) \to (\alpha \to \beta)) \to (\neg\, \alpha \to ((\neg\, \beta \to \neg\, \alpha) \to (\alpha \to \beta)))$ [Axiom 1]
3. $\neg\, \alpha \to ((\neg\, \beta \to \neg\, \alpha) \to (\alpha \to \beta))$ [MP 1, 2]
4. $(\neg\, \alpha \to (\neg\, \beta \to \neg\, \alpha)) \to (\neg\, \alpha \to (\alpha \to \beta))$ [Axiom 2 and MP]
5. $\neg\, \alpha \to (\neg\, \beta \to \neg\, \alpha)$ [Axiom 1]
6. $\neg\, \alpha \to (\alpha \to \beta)$ [MP 4, 5]

$\boxed{\text{Now, we know: } \vdash \neg\, \alpha \to (\alpha \to \beta)}$

by converse D.T

$\boxed{\neg\, \alpha \vdash \alpha \to \beta}$

by converse D.T

$\boxed{\{\neg\, \alpha, \alpha\} \vdash \beta}$

Remark: From a premise comprising of wffs α, $\neg\, \alpha$, any wff β follows.

This property is known as **explosiveness**.

4.3 Negation inconsistent and Absolute inconsistent

Definition 17 ***Negation inconsistency:*** *a set Γ is inconsistent iff there is a wff α such that $\Gamma \vdash \alpha$, $\Gamma \vdash \neg \alpha$.*

Remark: So, from a negation inconsistent set Γ, any β follows (see the above box).

Definition 18 ***Absolute inconsistency:*** *When, from an inconsistent premise Γ, any β follows.*

$$\Gamma \vdash \beta(any)$$

Trivially, from absolute inconsistency, negative inconsistency follows.

Exercise: Prove that: if $\Gamma \cup \{\neg \alpha\} \vdash \beta, \neg \beta$ then $\Gamma \vdash \alpha$.

Note:

- In classical logic, negation inconsistency is equivalent to absolute inconsistency but there are logics where this equivalence does not hold.
- Paraconsistent logic is a logic group which does not believe in this equivalence.
- Paraconsistency means, if $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg \alpha$, that does not imply $\Gamma \vdash \beta$, for all β. That is, they do not believe in that inconsistency implies explosiveness.
- "If $\Gamma \cup \{\neg \alpha\} \vdash \beta, \neg \beta$ then $\Gamma \vdash \alpha$." is reductio ad absurdum accepted by the intuitionist logician or constructivist.
- What about "If $\Gamma \cup \{\alpha\} \vdash \beta, \neg \beta$ then $\Gamma \vdash \neg \alpha$."?

The intuitionists do not accept this generally.

Chapter 5

Propositional Logic: Soundness and Completeness

Class 5: Dated 12 **- August -** 2016

According to Chapter 2 and 3, we can define consequence relation in two different way i.e. $\Gamma \vdash \alpha$ and $\Gamma \vdash_{A_x} \alpha$.

Definition 19 *$\Gamma \vdash \alpha$ means, when for all v, $v(\Gamma) = \{1\}$, then $v(\alpha) = 1$. Here v is valuation. In this relation we use valuation. So, it is called* **semantic definition**. *The motivation for such a definition is that it is about a valid argumentation.*

Definition 20 *Similarly, $\Gamma \vdash_{A_x} \alpha$ iff there is a sequence $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$ of wffs such that–*

(i) $\alpha_n = \alpha$;

(ii) Any α_i, is either in axiom(A_x) or in Γ or is obtained by the rule MP from the previous wffs of the sequence.

So the relation $\Gamma \vdash_{A_x} \alpha$ is based on only syntax. For that reason it is called **syntactic definition**. *For computer, we need this syntactic relation.*

Now, we have to see the relationship between these two types of definition $\Gamma \vdash \alpha$ and $\Gamma \vdash_{A_x} \alpha$.

Here, many types of situation may occur. (i) $\Gamma \vdash \alpha$ holds but machine cannot derive, i.e. $\Gamma \nvdash_{A_x} \alpha$, (ii) $\Gamma \vdash_{A_x} \alpha$ holds but $\Gamma \nvdash \alpha$ and (iii) both $\Gamma \vdash \alpha$ and $\Gamma \vdash_{A_x} \alpha$ holds.

Now we can say that in case of situation (ii), the program is not good enough and in case of situation (iii) the program is good. Situation (iii) can be shown by proving: $\Gamma \vdash_{A_x} \alpha \Rightarrow \Gamma \vdash \alpha$ and $\Gamma \vdash \alpha \Rightarrow \Gamma \vdash_{A_x} \alpha$.

*[**Note:** In physics, semantics (i.e. observations) and syntactic method of everything do not match]*

5.1 Soundness and Completeness

Definition 21 $\Gamma \vdash_{A_x} \alpha \Rightarrow \Gamma \vdash \alpha$

means, if the machinery produce α from Γ then $\Gamma \vdash \alpha$ holds This system is called **sound**.

Proof of Soundness is obtained by,
first showing that all the axioms are tautologies and
second, the rule M.P preserves truth i.e. if for any valuation v, $v(\alpha) = 1$ and $v(\alpha \to \beta) = 1$ then $v(\beta) = 1$.

Now, let $\Gamma \vdash_{A_x} \alpha$. Then there is a chain,

$$\alpha_1$$

$$\alpha_2$$

$$.$$

$$.$$

$$.$$

$$\alpha_n \ (\equiv) \ \alpha$$

With the conditions stated in Definition 14. So, if for a valuation v, $v(\Gamma) = \{1\}$, then $v(\alpha_n) = v(\alpha) = 1$.

Definition 22 $\Gamma \vdash \alpha \Rightarrow \Gamma \vdash_{A_x} \alpha$

means, when $\Gamma \vdash \alpha$ holds then the machinery produce each α for any Γ, this is known as **complete**.

[Note that, Any observation is ***semantic*** but the explanation is ***syntactic***. It is also possible that *Syntax may or may not be satisfy Semantics*.

Example 17 As an example, we can write that Physical theory = Syntax, Practical experiment = Semantic.]

For proof of Completeness, see Chapter 8

Chapter 6

Modal Propositional Logic

Class 5: Dated 12 **- August -** 2016

6.1 Modal Propositional Logic

Modal logic can be defined by the pair ($\mathcal{L}$,$\vdash_{A_x}$). Here, the alphabet $\mathcal{A}$ of $\mathcal{L}$ is extended, $\mathcal{A}$ = {p,|,∼,→,(,),□}. Note that, the extended part is □, where □ is a unary logical connective.

Definition 23 *The object language $\mathcal{L}$ over the alphabet A is defined as follows.*

1. *Any p followed by a finite number of | marks in $\mathcal{L}$.*
2. *If α is a wffs then $(\sim \alpha)$, $(\Box\alpha)$ are formulas.*
3. *If α, β are wffs then $(\alpha \to \beta)$ is a wffs.*
4. *Nothing else is a wff.*

Example 18 $\Box p \mid$, $\Box(\sim p \mid)$ are wffs but $(p \mid \Box p \mid\mid)$ is not a wff.

• **Necessity Operator** (□): Unary logical connectives □ is known as **necessity operator**. We read $\Box p \mid$ as 'necessarily $p \mid$', where $p \mid$ is a sentence.

$p \mid$: Ram is honest.
$\sim p \mid$: Ram is not honest.
$\Box \sim p \mid$: Necessarily Ram is not honest.
$\underline{\sim \Box \sim}\, p \mid$: *<u>It is not that necessarily Ram is not honest.</u>*

$\Updownarrow$ $\Updownarrow$

$\Diamond$ **possibly** Ram is honest.

So here another logical connectives is derived, i.e. $\sim \Box \sim \equiv \Diamond$, which is known as **possibility operator**($\Diamond$).

•**Possibility Operator**($\Diamond$): we read $\Diamond p \mid$ as possibly $p \mid$, where $p \mid$ is a sentence.

$p \mid$: Ram is honest.
$\sim p \mid$: Ram is not honest.
$\Diamond \sim p \mid$: Possibly Ram is not honest.

$\underline{\sim \Diamond \sim}\ p \mid$: <u>*It is not that possibly Ram is not honest.*</u>

$\Updownarrow$ $\Updownarrow$

$\Box$ **necessarily** Ram is honest.

So we can also drive $\Box$ from $\Diamond$, i.e. $\sim \Diamond \sim \equiv \Box$.

Note that, all the tautologies are necessarily true. That does not mean only tautologies are necessarily true. So, Tautology $\subseteq$ Necessarily True.

For example, 'the angle sum of triangles is two right angles' is necessarily true but not a tautology.

6.2 Accessibility Relation

A set of situations W $\equiv \{W_1, W_2, \cdots\}$ which is shown in Figure 6.1. Consider a binary relation R in W i.e, between two such situation either W_1 R W_2 holds or does not hold. Now if we again consider that in W_1, p_1, p_2, p_3 and p_4 are four different sentences. In W_1 the value of those sentences are shown in figure 6.1. The value of those sentences may be different in situation W_2, i.e. $\mathrm{V}(p_1, W_1) = 1$ and $\mathrm{V}(p_1, W_2) = 0$. Set of situations with a binary relation i.e. (W, R) is known as **Kripke frame**.

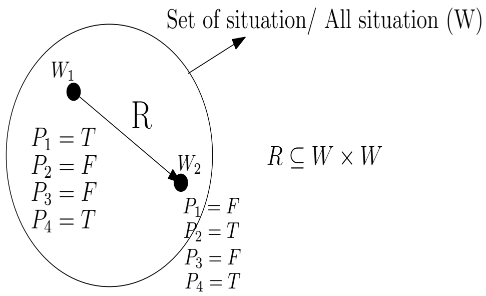


Figure 6.1: Set of situations

In case of classical propositional logic the valuation v: $L \to \{1, 0\}$ but in case of modal propositional logic v: $L \times W \to \{1, 0\}$. In modal propositional logic, the valuation function can be extended over the whole set of wffs by,

- $$v(\sim \alpha, W_i) = \begin{cases} 1 & \text{iff } v(\alpha, W_i) = 0 \\ 0 & \text{iff } v(\alpha, W_i) = 1 \end{cases}$$

- $$v(\alpha \to \beta, W_i) = \begin{cases} 0 & \text{iff } v(\alpha, W_i) = 1 \text{ and } v(\beta, W_i) = 0 \\ 1 & otherwise \end{cases}$$

- $$v(\alpha \wedge \beta, W_i) = \begin{cases} 1 & \text{iff } v(\alpha, W_i) = 1 \text{ and } v(\beta, W_i) = 1 \\ 0 & otherwise \end{cases}$$

- $$v(\alpha \vee \beta, W_i) = \begin{cases} 1 & \text{iff } v(\alpha, W_i) = 1 \text{ or } v(\beta, W_i) = 1 \\ 0 & otherwise \end{cases}$$

- $v(\Box\alpha, W_i) = 1$ iff for all W' such that WRW' holds we get $v(\alpha, W') = 1$ (Necessarily α is true in W, if α is true at all situations related to W.

- $v(\Diamond\alpha, W) = 1$ iff for some W' such that WRW' holds we get $v(\alpha, W') = 1$

Note that, W' is said to be accessible from W. For that reason the relation R is known as ***accessibility relation***.

6.3 Modal System T

A modal system T with axioms and rules:

Axioms:

- *PL Axioms:* Propositions logics axioms.
- Proper modal axioms:
  - *Axioms K:*
    $$\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$$
    Therefore, if necessarily $(\alpha \to \beta)$ and necessarily α then necessarily β.
  - *Axiom T:*
    $$\Box\alpha \to \alpha$$
    Therefore, if necessarily α then α is true or if something is necessarily true then it is true.

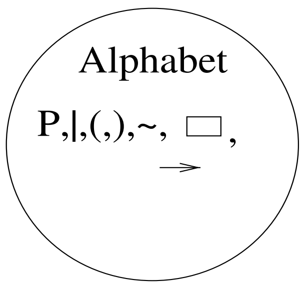
Note: Another interpretation of $\Box\alpha$ may be 'one knows that α'. According to this interpretation Axiom K says 'if one knows that $(\alpha \to \beta)$ and knows that α then one knows that β', this is called forward (positive) introspection. Under this interpretation Axiom T reads as 'if one knows that α then α (is true)'.

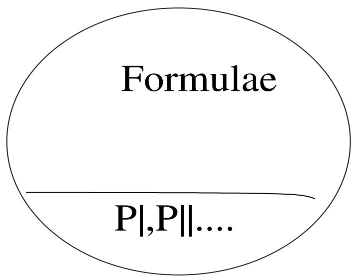
Rules:

- *MP:* $\dfrac{\alpha, \alpha \to \beta}{\beta}$
- *N:* $\dfrac{\alpha}{\Box\alpha}$ [if α is true everywhere then α is necessarily true.]

In the interpretation, we consider reflexive relation. When we change the relation then axiom T changes, but other axioms and rules are fixed.

Here, to the alphabet of propositional logic, one extra operator $\Box$ was added, which is an unary operator.





Now, the question is: How is formulae $\mathcal{L}$ formed? As in Def. 23,

- P$|\cdots|$ (n times, where $n \geq 1$), i.e. P$|$,P$||$,$\cdots$.
- If α is a wff, then $\backsim \alpha, \Box\alpha$ are also wffs.
- If α, β are wffs, then $(\alpha \to \beta)$ is a wff.
- Nothing else.

Here, $\mathcal{L}$ is the language. Now, from a different direction, any operator in propositional logic is identified by the truth table.

α	$\backsim \alpha$
T	F
F	T

Note that, this is a mapping $\backsim$: {T, F} $\mapsto$ {T, F}. Now, we try to understand the truth value of $\Box$.

α	$\Box\alpha$
T	?
F	?

Here, the question is that if α is T or F, can we say what would be the value of $\Box\alpha$?

Let us observe that, if in a formula, there are n atomic formulae, then it forms an n-ary function in case of classical propositional logic. As an example, (P1 $\rightarrow\backsim$ P2) $\wedge$ P3 gives the function $\{T,F\}^3 \rightarrow \{T,F\}$. Therefore, the corresponding table can be shown as follows:

P1	P2	P3	(P1 $\rightarrow\backsim$ P2) $\wedge$ P3
...	...	...	...

Technically the function is written as, $f_{(P1\rightarrow\backsim P2)\wedge P3}$.

Similarly, for n-atomic formulae, the function can be written as $\{T,F\}^n \rightarrow \{T,F\}$.

Now, let us assume an arbitrary function $\{T,F\}^3 \rightarrow \{T,F\}$:

P1	P2	P3	?
T	T	T	F
T	T	F	T
T	F	T	T
T	F	F	F
F	T	T	F
F	T	F	F
F	F	T	F
F	F	F	F

Here, as there is no formula mentioned but after the table is made, obviously this becomes a function. Such a function is known as <u>truth function</u>.

Now given any truth function, a question may arise: does there exist any formula whose truth function is the given one ? The answer is 'Yes'.

Note: An analogy can be drawn in this context when we try to find such formula. It is the Karnaugh map. A Karnaugh map is a method of simplifying Boolean algebra expressions. For ease of understanding let us take an example given in the following truth table. A formula is written in a disjunctive normal form according to the truth values of the table. The formula is, $(P1 \wedge \backsim P2 \wedge P3) \vee (P1 \wedge \backsim P2 \wedge \backsim P3) \vee (\backsim P1 \wedge P2 \wedge P3) \vee (\backsim P1 \wedge \backsim P2 \wedge \backsim P3)$. This formula can be minimized with the help of a Karnaugh map (K-map) as shown in Fig. 6.2. The minimized form of the formula is, $(\backsim P2 \wedge \backsim P3) \vee (P1 \wedge \backsim P2) \vee (\backsim P1 \wedge P2 \wedge P3)$

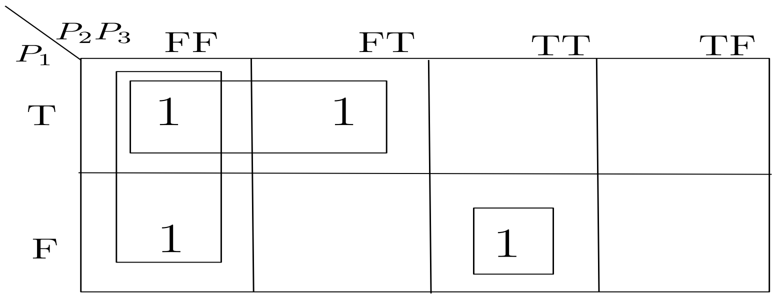


Figure 6.2: Karnaugh Map

Example 19 Let us take a truth function,

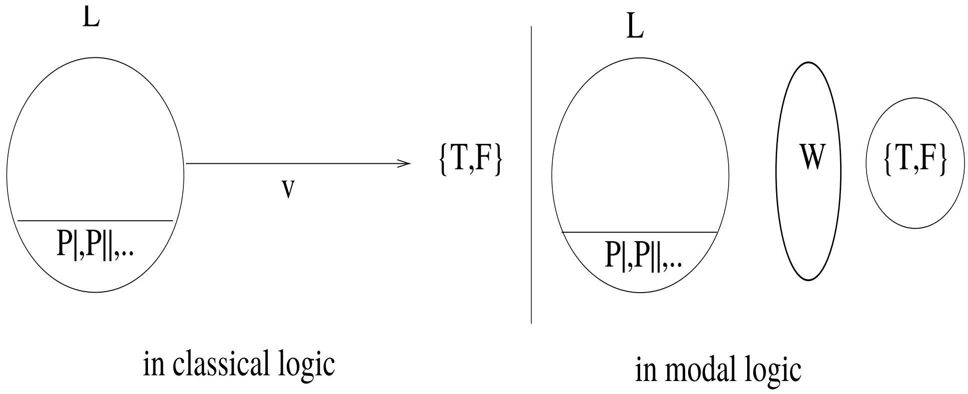
P1	P2	P3	?	
T	F	T	T	$(P1 \wedge \backsim P2 \wedge P3)$
T	F	F	T	$(P1 \wedge \backsim P2 \wedge \backsim P3)$
F	T	T	T	$(\backsim P1 \wedge P2 \wedge P3)$
F	F	F	T	$(\backsim P1 \wedge \backsim P2 \wedge \backsim P3)$

Therefore, the formula is $(P1 \wedge \backsim P2 \wedge P3) \vee (P1 \wedge \backsim P2 \wedge \backsim P3) \vee (\backsim P1 \wedge P2 \wedge P3) \vee (\backsim P1 \wedge \backsim P2 \wedge \backsim P3)$.

Any of the pairs $\{\wedge, \vee\}$, $\{\vee, \backsim\}$, $\{\rightarrow, \backsim\}$ is enough to express any truth function. This result is known as **Adequacy theorem**. So, extra unary or binary operator taken would be redundant in classical logic. Hence, we can not interprete $\Box$ operator in this way. Now, we will discuss about the modal logic.

6.4 Entering into Modal logic:

Let us recall that, in modal logic, the valuation can be written as v: P $\times$ W $\rightarrow$ {T,F}, where P is the set of propositional variables. Hence, we can write v(P1,W_1) = T/F.



Definition 24 *$v(\backsim \alpha, W_i) = T$ iff $v(\alpha, W_i) = F$.*

It is read as, $v(\backsim \alpha)$ at W_i is true if and only if v(α) at W_i is false.

Definition 25

$$v(\alpha \to \beta, W_i) = \begin{cases} F & \textit{iff } v(\alpha, W_i) = T \textit{ and } v(\beta, W_i) = F \\ T & \textit{Otherwise} \end{cases}$$

It follows,

1. $v(\alpha \wedge \beta, W_i)$ = T iff $v(\alpha, W_i)$ = T and $v(\beta, W_i)$ = T.
2. $v(\alpha \vee \beta, W_i)$ = T iff $v(\alpha, W_i)$ = T or v(β, W_i) = T.

Now, our issue is when we can say $v(\Box\alpha, W_i)$ = T.

To address this, let us consider, W: non-empty set of worlds; and R: binary relation on W. Therefore, we get (W,R) and W_1 R W_2 may or may not hold for W_1, $W_2 \in W$. W_1RW_2 means W_2 is accessible to W_1.

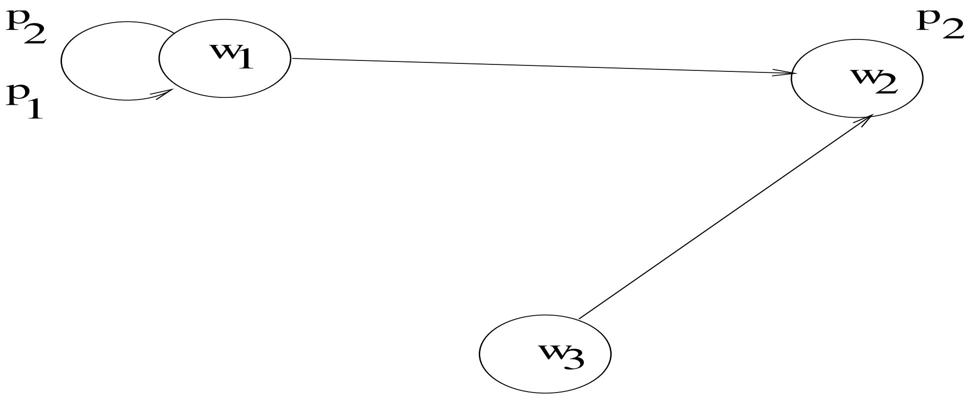
Definition 26 *$v(\Box\alpha, W_i)$=T iff $v(\alpha, W') = T$, for all W' such that $W_i R W'$ holds. Here, R is called the accessibility relation.*

Example 20 Let us consider, W = $\{W_1, W_2, W_3\}$ and R = $W_1RW_2, W_1RW_1, W_3RW_2$.

(Given that P1, P2 are true in W_1. P2 is true in W_2 at W_j. Let us take a formula $P_1 \to \Box P_2$. We want to calculate its value, i.e. $v(P_1 \to \Box P_2, W_1)$ = ?

Here, P_1,P_2 are true in W_1, and P_2 is true in W_2. Related to this, we want to calculate the value $v(P_1 \to \Box P_2, W_1)$. Now, $v(P_1 \to \Box P_2, W_1)$ = T when

1. $v(\Box P_2, W_1)$ = T, $v(P_1, W_1)$ = T.

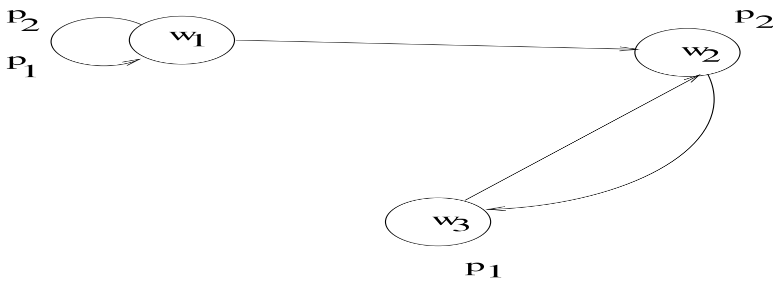


2. $v(\Box P_2, W_1) = \text{T}$, $v(P_1, W_1) = \text{F}$.

3. $v(\Box P_2, W_1) = \text{F}$, $v(P_1, W_1) = \text{F}$.

Here, $v(P_1, W_1) = \text{T}$ and $v(\Box P_2, W_1) = \text{T}$. Hence $v(P_1 \to \Box P_2, W_1) = \text{T}$.

Example 21 Let us take another example.



Evaluate (i) $v(P_1 \to \Box P_2, W_2)$ (ii) $v(P_2 \to \Box P_2, W_2)$ (iii) $v(\Box\Box P_1, W_2)$.

(i) Here, $v(P_1, W_2) = \text{F}$ and $v(\Box P_2, W_2) = \text{F}$. Hence $v(P_1 \to \Box P_2, W_2) = \text{T}$.

(ii) Here, $v(P_2, W_2) = \text{T}$ and $v(\Box P_2, W_2) = \text{F}$. Hence $v(P_2 \to \Box P_2, W_2) = \text{F}$.

(iii) $v(\Box\Box P_1, W_2) = \text{T}$ iff $v(\Box P_1, W_2') = \text{T}$ for all W_2'. Here, W_2' is related to W_3. P_1 is here in W_3. Hence $v(\Box P_1, W_2') = \text{T}$. Therefore $v(\Box\Box P_1, W_2) = \text{T}$.

Note:

1. If R is reflexive, T-axiom is always true, i.e. $\Box\alpha \to \alpha$.

2. The issue of decidability in case of classical logic is: "Whether the formula is satisfiable or not". However, the issue of decidability in case of modal logic is: "If it is satisfiable, then can we construct W and R ?"

3. $\Box$ can also be interpreted as knowledge operator. In case of knowledge operator $\Box\Box P_1 \equiv \Box P_1$. In knowledge operator $\Box\alpha \to \alpha$ means "If I know α, then α is true".
4. $\Box P_1 \vee \backsim \Box P_1$ is a classical tautology.
5. $(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ is a modal tautology.
6. Kripke frame is a network.

Dated 12 - September - 2017

The alphabet in Modal propositional logic consists of: p, $|$, $\neg$, $\to$, $\Box$,), (. With the help of these symbols, one forms the following wffs:
$p|$, $p||$, $\cdots$
p_1, p_2, $\cdots$
$(\neg\alpha)$, $(\alpha \to \beta)$, $(\Box\alpha)$

The axioms and rules of modal propositional logic are given as follows.
Axioms: All Propositional calculas (PC) axioms
+
K: $(\Box(\alpha \to \beta)) \to (\Box\alpha \to \Box\beta)$
D: $\Box\alpha \to \Diamond\alpha$
T: $\Box\alpha \to \alpha$
B: $\alpha \to \Box\Diamond\alpha$
S_4: $\Box\alpha \to \Box\Box\alpha$
S_5: $\Diamond\Box\alpha \to \Box\alpha$
Rules: $\frac{\alpha, \alpha\to\beta}{\beta}$, M.P
$\frac{\alpha}{\Box\alpha}$, N.

Axioms **K**, **D**, **T**, **B**, **S_4** and **S_5** are additional axioms. All PC axioms and addional axiom **K** together constitute the minimal modal system. If axiom **B** is derived, axioms **K**, **D** and **T** will automatically be followed. The hierarchy of modal systems is given by System **K**, System **KT**, System **KTB**, System **KT**S_4, System **KT**S_5. Fig. 6.3 depicts this hierarchy.

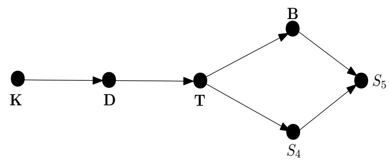


Figure 6.3: Graph representing the hierarchy order of modal logic systems

$\wedge$ and $\vee$ can be defined in terms of $\neg$ and $\rightarrow$. For example,
$\alpha \wedge \beta \equiv \neg(\alpha \rightarrow \neg\beta)$
$\alpha \vee \beta \equiv (\neg\alpha) \rightarrow \beta$.

$\Diamond\alpha$ can be defined in terms of $\Box$ as follows.
$\Diamond\alpha \equiv \neg(\Box(\neg\alpha))$
It means that 'it is not the case that $\neg\alpha$ is necessary'. That is, we can say, α is possible. This symbol $\Diamond \equiv$ *possible*.
Hence, all the operators are, $\neg, \rightarrow, \vee, \wedge, \Box, \Diamond$.

- **D**-axiom: $\Box\alpha \rightarrow \Diamond\alpha$

Proposition 1 : *Axiom* ***D*** *can be derived from* ***T***

Proof : Assume **T**.
So, $\Box\alpha \rightarrow \alpha$. We now derive D.
1. $\Box\neg\alpha \rightarrow \neg\alpha$ AxT
2. $\neg\neg\alpha \rightarrow \neg\Box\neg\alpha$ PC
3. $\alpha \rightarrow \neg\neg\alpha$ PC
4. $\alpha \rightarrow \neg\Box\neg\alpha$ HS
5. $\Box\alpha \rightarrow \alpha$ T
6. $\Box\alpha \rightarrow \neg\Box\neg\alpha$ HS
7. $\Box\alpha \rightarrow \Diamond\alpha$

$\Box$

In **T**, $\Box\alpha \rightarrow \alpha$ and $\alpha \rightarrow \Diamond\alpha$ are available. If $\Box\alpha \rightarrow \alpha$ is an axiom, then $\alpha \rightarrow \Diamond\alpha$ is a dual axiom. That is, if it is necessary then it is possible.

Exercise: Prove that $\Box\alpha \equiv \neg\Diamond\neg\alpha$

Note: From **D** one can not derive **T**, but from **T**, one can derive **D**.

6.5 Problems of Material Implication:

In the history of formation of modal logic, the following problems arose. Among the four cases of implication, (T, F) $\rightarrow$ F is easily acceptable by everyone. However, the other cases $(F, T) \rightarrow T$ and $(F, F) \rightarrow T$ are not easily accepted by common sense. For example,
If $2 > 3$**, then** $2 + 3 = 5$. By definition it is true. But from intuition, it is not acceptable. Again,
If $2 > 3$**, then** $2+3 = 6$. By definition, this is also T. But this is also not easily accepted intuitively. Such problems again arise in material implication. Because

of these implication $\rightarrow$ seems paradoxical. For example, $(\alpha \rightarrow \beta) \vee (\beta \rightarrow \alpha)$ is a tautology. This means either $\alpha \rightarrow \beta$ or $\beta \rightarrow \alpha$. These are called problems of material implications.

Because of these problems in implication, implication should be more strict (Lewis). Therefore, a new concept, called "strict implication" is formed and it is denoted by '$\prec$'. Now modal logic can be defined as,

Definition 27 $\alpha \prec \beta \equiv \Box(\alpha \rightarrow \beta)$

So, in the enhanced modal logic there are two implication:

1. $\rightarrow$ (material implication)
2. $\prec$ (strict implication)

'$\prec$' can imply '$\rightarrow$' i.e.
$\frac{\alpha \prec \beta}{\alpha \rightarrow \beta}$.

Proof : 1. $\Box(\alpha \rightarrow \beta)$
2. In system **T**, we have $\Box(\alpha \rightarrow \beta) \rightarrow (\alpha \rightarrow \beta)$
3. $\alpha \rightarrow \beta$, MP 1, 2 $\Box$

$\Box$ ditributes over conjunction.

$\boxed{\vdash_T \Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)}$

As, $\Box$ distributes, its dual property can be written as,

$\boxed{\vdash_T \Diamond(\alpha \vee \beta) \leftrightarrow (\Diamond\alpha \vee \Diamond\beta)}$

6.6 Semantics:

Truth table for $\Box\alpha$ is meaningless, adequacy theorem says, any truth function can be obtained by using $(\neg, \vee)$ or $(\neg, \wedge)$ or $(\neg, \rightarrow)$. So, there is no need to form truth table for $\Box\alpha$ for its semantics.

Kripke Semantics:

Valuation $v : P \times W \rightarrow \{T, F\}$ where P is the set of propositional variables and W is a non-empty set, called the set of worlds. $v(p_i, w_i) = T/F$. For any α,

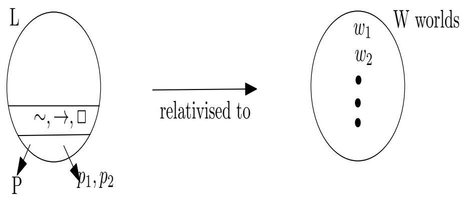


Figure 6.4

we need an extension. Now we extend v over L obeying the following rule:

$$v(\neg\alpha, w) = \begin{cases} T & \text{iff } v(\alpha, w) = \text{F} \\ F & \text{iff } v(\alpha, w) = \text{T} \end{cases} \tag{6.1}$$

$$v(\alpha \to \beta, w) = \begin{cases} F & \text{iff } v(\alpha, w) = T \text{ and } v(\beta, w) = F \\ T & \text{Otherwise} \end{cases} \tag{6.2}$$

Semantics for the other connectives $\vee$ and $\wedge$ follow.

$$v(\alpha \wedge \beta, w) = \begin{cases} T & \text{iff } v(\alpha, w) = T \text{ and } v(\beta, w) = T \\ F & \text{Otherwise} \end{cases} \tag{6.3}$$

$$v(\alpha \vee \beta, w) = \begin{cases} T & \text{iff } v(\alpha, w) = T \text{ or } v(\beta, w) = T \\ F & \text{Otherwise} \end{cases} \tag{6.4}$$

To give semantics for $\Box\alpha$, we need a binary relation R on W. The relation is called *Accessibility relation*.

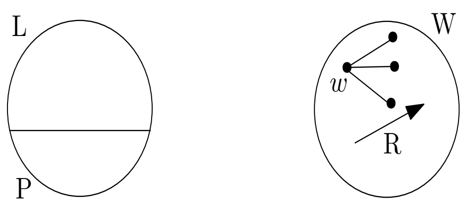


Figure 6.5

$v(\Box\alpha, w) = T$ iff $v(\alpha, w') = T$ for all w' such that wRw' holds. It follows that,

$$v(\Diamond\alpha, w) = T\text{iff } v(\alpha, w') = T \text{ for some } w' \text{ s.t } wRw' \tag{6.5}$$

Exercise: Check that we don't need any condition on R to show that axiom **K** is true at all worlds.

We have the rules MP and N.

$$\frac{\alpha, \alpha \to \beta}{\beta}, \text{ M.P}$$

$$\frac{\alpha}{\Box\alpha}, \text{ N.}$$

Now, we can derive two new rules using PC axioms and axiom K:

Proposition 2 : DR_1**:** $\dfrac{\alpha \to \beta}{\Box\alpha \to \Box\beta}$

Proof :

1. $\alpha \to \beta$
2. $\Box(\alpha \to \beta)$ N
3. $\Box(\alpha \to \beta) \to (\Box\alpha \to \Box\beta)$ Axiom K
4. $\Box\alpha \to \Box\beta$ MP

$\Box$

Proposition 3 : DR_2**:** $\dfrac{\alpha \to \beta}{\Diamond\alpha \to \Diamond\beta}$

Proof :

1. $\alpha \to \beta$
2. $(\alpha \to \beta) \to (\neg\beta \to \neg\alpha)$ PC
3. $\neg\beta \to \neg\alpha$ MP
4. $\Box\neg\beta \to \Box\neg\alpha$ DR_1
5. $\neg\Box\neg\alpha \to \neg\Box\neg\beta$ PC
6. $\Diamond\alpha \to \Diamond\beta$ Definition of $\Diamond$

$\Box$

Now, we can prove the following proposition in system K.

Proposition 4 : $\vdash_K \Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$

Proof : Proof of $\vdash_K \Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta)$:

1. $(\alpha \wedge \beta) \to \alpha$ PC

2. $\Box(\alpha \wedge \beta) \to \Box\alpha \qquad DR_1$

3. $(\alpha \wedge \beta) \to \beta \qquad$ PC

4. $\Box(\alpha \wedge \beta) \to \Box\beta \qquad DR_1$

5. $\Box(\alpha \wedge \beta) \to (\Box\alpha \wedge \Box\beta) \qquad$ PC: $\dfrac{\alpha \to \beta, \quad \alpha \to \delta}{\alpha \to (\beta \wedge \delta)}$

Similarly, we can also prove $\vdash_T (\Box\alpha \wedge \Box\beta) \to \Box(\alpha \wedge \beta)$.
So, by PC, $\vdash_K \Box(\alpha \wedge \beta) \leftrightarrow (\Box\alpha \wedge \Box\beta)$ $\Box$

6.7 System K, Semantics

In system K, axiom K and all PC axioms are axioms of this system (see Figure 6.6). Now, take a Kripke Frame $\langle W, R\rangle$ and a valuation v such that

$$v : P \times W \to \{T, F\}$$

Note that, a Kripke Frame with a valuation v is a model.

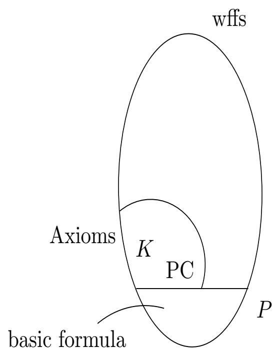


Figure 6.6: System K

- A wff α is true in a model $\langle W, R, v\rangle$, if and only if, $v(\alpha, w) = T$ for all $w \in W$.

Example 22 Take $W = \{w_1, w_2\}$, where $w_1 R w_2$ and $w_2 R w_2$. α is a wff with only p_1, p_2. The valuation function v is defined as follows:

$$v(p_1, w_1) = T$$

$$v(p_2, w_1) = F$$

$$v(p_1, w_2) = T$$

$$v(p_2, w_2) = T$$

By using this, $(\Box p_1, w_1) = T$. Therefore, in this way, all formulas can be assigned a truth value at each world.

- A formula is true in a model, if it is true in all worlds.
- A formula is universally true or valid, if and only if, it is true in all models. That is, for all W, for all R, for all v, the formula has to be true.

Proposition 5 : *All axioms of System K are valid/universally true.*

Proof :

PC: Take an arbitrary axiom, say $\alpha \to (\beta \to \alpha)$. Now, pick any world w and any valuation v obtains $v(\alpha)$ and $v(\beta)$which are T/F. However, we know that, $\alpha \to (\beta \to \alpha)$ is a tautology. So, it is valid irrespective of $\langle W, R, v\rangle$. Similarly, other axioms of PC can be proved to be valid.

Ax. K: We know, axiom K is $(\Box(\alpha \to \beta)) \to (\Box\alpha \to \Box\beta)$.

Take an arbitrary model $\langle W, R, v\rangle$. Take any $w \in W$. Now, the question is

$$v(K, w) =?$$

Let, it be F. This implies,

$$v((\Box(\alpha \to \beta)), w) = T \text{ and } v(\Box\alpha \to \Box\beta), w) = F$$

$$\text{That is, } v((\Box(\alpha \to \beta)), w) = T \text{ and } v(\Box\alpha, w) = T \text{ and } v(\Box\beta, w) = F$$

Now, $v(\Box\alpha, w) = T \Leftrightarrow v(\alpha, w') = T$ for all w', such that wRw'; $v(\Box\beta, w) = F \Leftrightarrow v(\beta, w'') = F$ for some w'', such that wRw''; and $v((\Box(\alpha \to \beta)), w) = T \Leftrightarrow v(((\alpha \to \beta)), w') = T$ for all w', such that wRw'.

Therefore, the situation is, at w'', α is T and β is F, so, $\alpha \to \beta$ is F. But, this contradicts $v((\Box(\alpha \to \beta)), w) = T$. So, $v(K, w) = T$.

□

6.8 System T

In system T, axiom K, axiom T and all PC axioms are axioms of this system (see Figure 6.7). Now, take a Kripke Frame $\langle W, R\rangle$ and a valuation v.

Proposition 6 : *The axiom $T : \Box\alpha \to \alpha$ is valid/universally true in all reflexive frames.*

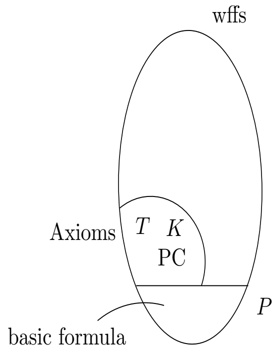


Figure 6.7: System T

Proof : For the axiom $T : \Box\alpha \to \alpha$ to be valid/universally true, $v(T, w)$ must be true for all $w \in W$. That is, $v(\Box\alpha \to \alpha, w) = T$. We shall show that, it can not be F.

For, if F, $v(\Box\alpha, w) = T$ and $v(\alpha, w) = F$. This situation can not be possible, if we put a restriction on R. Take R as reflexive, then $v(\Box\alpha, w) = T$ and $v(\alpha, w) = F$ is not possible since wRw holds. That means, if R is reflexive, axiom T is valid. $\Box$

Reflexivity of R is a sufficient condition for axiom T to be valid. But, is this also a necessary condition? We can prove that, this is also a necessary condition. Therefore, system T is valid on all models with reflexive relations. The Kripke Frame associated with system T is called reflexive Kripke Frame.

6.9 Other Axioms

Proposition 7 : *The axiom S_4 is true in all transitive models.*

Proof : We have to prove that, $v(S_4, w) = T$ if R is transitive. That is, $v(\Box\alpha \to \Box\Box\alpha, w) = T$ for all w.

Let $v(\Box\alpha \to \Box\Box\alpha, w) = F$. This implies, $v(\Box\alpha, w) = T$ and $v(\Box\Box\alpha, w) = F$.

Now, a possible situation is shown in Figure 6.8. Here, the relation R is shown by the arrows. As, R is transitive, so, wRw_5. As, α is F in w_5, so it contradicts $v(\Box\alpha, w) = T$. Therefore, $v(\Box\alpha \to \Box\Box\alpha, w)$ can not be F. Hence, $v(S_4, w) = T$ if R is transitive. $\Box$

Models of S_4 are reflexive and transitive frames.

Proposition 8 : *The axiom B is true in all symmetric models.*

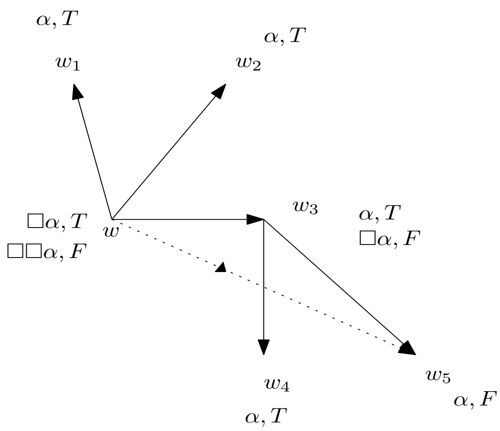


Figure 6.8

This can be similarly established that $v(\alpha \to \Box\Diamond\alpha) \neq F$ by showing a situation in which R is symmetric.

Therefore, for system B, we need reflexive and symmetric frames. It follows that, for system S_5, models have to be reflexive, symmetric and transitive – that is equivalent models with equivalence relations.

<u>**Note:**</u> In computer science modal operators are used to deal with many notions other than necessity and possibility. For example, the operator $\Box$ is used widely as knowledge operator and modal systems S_4 and S_5 are usually taken as modal systems appropriate for the linguistic phrase 'know that'.

Chapter 7

Boolean Algebra and Propositional Logic

Class 6: Dated 30 **- August -** 2016

7.1 Equivalence Relation

Definition 28 ***Equivalence Relation:*** *A relation $\mathcal{R}$ on a set X is said to be an equivalence relation if the relation $\mathcal{R}$ is reflexive, symmetric and transitive.*

The equivalence relation $\mathcal{R}$ partitions the set X into non-empty disjoint subsets. In particular the partition may consist of only the whole set X.

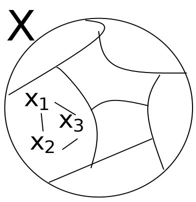


Figure 7.1: Set X is divided into several subsets

Consider Figure 7.1 where a set X is divided into several subsets. Say, some elements x_1, x_2, x_3 exist in one of the subsets and are in equivalence relation $\mathcal{R}$ with each other.

The **Equivalence class** of an element x is $= \{x | x' \mathcal{R} x\}$.

Note that. a equivalence class of x is denoted by $[x]_{\mathcal{R}}$ or $[x]$ simply.

Definition 29 ***Quotient Set:*** $\mathcal{X}/\mathcal{R} = \{[x] \; | x \in \mathcal{X}\}$. *Therefore, Quotient set is a class of subsets of the set X.*

Now in the set $\mathcal{L}$ of wffs of propositional logic we define an equivalence relation.

Definition 30 *$\alpha\mathcal{R}\beta$ is holds if and only if $\vdash_{A_x} \alpha \to \beta$ and $\vdash_{A_x} \beta \to \alpha$.*

Theorem 4 *$\mathcal{R}$ is reflexive, symmetric and transitive, i.e. $\mathcal{R}$ is an equivalence relation.*

Sketch of proof:

- **Reflexivity:** We have proved earlier that, $\vdash_{A_x} \alpha \to \alpha$. So $\alpha\mathcal{R}\alpha$ holds.
- **Symmetry:** Let us assume $\alpha\mathcal{R}\beta$ holds. From definition we get $\vdash_{A_x} \alpha \to \beta$ and $\vdash_{A_x} \beta \to \alpha$. So, $\beta\mathcal{R}\alpha$ holds.
- **Transitivity:** Let $\alpha\mathcal{R}\beta$ and $\beta\mathcal{R}\gamma$ hold. Therefore, from definition we get

$$\vdash_{A_x} \alpha \to \beta \tag{7.1}$$

$$\vdash_{A_x} \beta \to \alpha \tag{7.2}$$

$$\vdash_{A_x} \beta \to \gamma \tag{7.3}$$

$$\vdash_{A_x} \gamma \to \beta \tag{7.4}$$

We need to show that $\alpha\mathcal{R}\gamma$ i.e. $\vdash_{A_x} \alpha \to \gamma$ and $\vdash_{A_x} \gamma \to \alpha$.

From equation 7.1 and 7.3, we get $\vdash_{A_x} \alpha \to \gamma$. Again from equation 7.2 and 7.4, $\vdash_{A_x} \gamma \to \alpha$. Therefore, $\alpha\mathcal{R}\gamma$ holds.

Note that, if $\mathcal{L}$ is a set and $\alpha \in \mathcal{L}$, then the quotient set is written as $\mathcal{L}/\mathcal{R}$: $\{[\alpha] \mid \alpha \in \mathcal{L}\}$

Definition 31 *Corresponding to operators $\wedge$, $\vee$ and $\sim$ in $\mathcal{L}$, we define operators $\bar{\wedge}$, $\underline{\vee}$ and $\neg$ in $\mathcal{L}/\mathcal{R}$ as follows:*

1. *$[\alpha] \mathbin{\bar{\wedge}} [\beta] = [\alpha \wedge \beta]$*
2. *$[\alpha] \mathbin{\underline{\vee}} [\beta] = [\alpha \vee \beta]$*
3. *$\neg[\alpha] = [\sim \alpha]$*

$\bar{\wedge}$, $\underline{\vee}$, $\neg$ are the new operators defined in quotient algebra.

7.2 Boolean Algebra

Definition 32 *Boolean Algebra is a* ***complemented distributive lattice****.*

7.2.1 Lattice

Definition 33 *A Lattice is a partially ordered set such that any pair of element has a least upper bound (lub) and greatest lower bound (glb).*

Partially Ordered Set:

Definition 34 *A set with a relation which is reflexive, anti-symmetric and transitive is known as Partially Ordered Set.*

In Figure 7.2, there are 8 elements in a set and 5 ordered chains where the relation is reflexive, anti-symmetric and transitive.

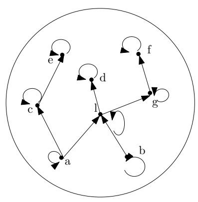


Figure 7.2: A Partially ordered set with 5 ordered chains

Least upper bound (lub) and Greatest lower bound (glb):

In Figure 7.2, l, d, f are the upper bounds of the elements a and b but the lub of elements a and b is l. lub of one element is the element itself. glb of elements e and d is a.

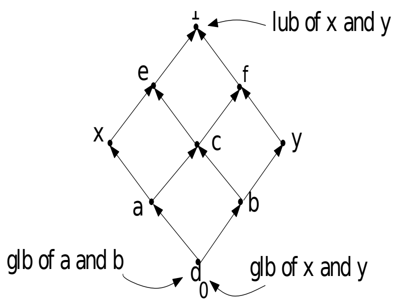


Figure 7.3: A Lattice

Consider Figure 7.3. A lattice must be a partially ordered set i.e there must be ordered chains. This figure has ordered chains. A lattice must have lub and glb of any two elements. Elements a and b have their lub c and glb d. Similarly elements x and c has its lub e and glb a. Thus, there is an lub and a glb for every two elements. So, Figure 7.3 is a lattice. An lub of two elements a and b is written as a $\veebar$ b and glb of two elements a and b is written as a $\barwedge$ b.

7.2.2 Distributive

Definition 35 *The following properties must hold for a lattice to be distributive.*

- $a \barwedge (b \veebar c) = (a \barwedge b) \veebar (a \barwedge c)$
- $a \veebar (b \barwedge c) = (a \veebar b) \barwedge (a \veebar c)$

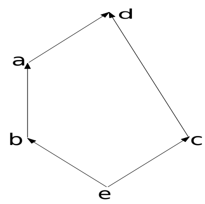


Figure 7.4: A Lattice but not distributive

Lattice but not distributive: Figure 7.4 shows the elements a, b and c do not satisfy the properties of distributivity but all the elements satisfy the properties of lattice. In this case, a $\barwedge$ (b $\veebar$ c) $\neq$ (a $\barwedge$ b) $\veebar$ (a $\barwedge$ c). So, Figure 7.4 is a lattice but not distributive .

7.2.3 Complementation

- The lattice should be bounded. There must be a top element and a bottom element. A finite lattice is a bounded lattice. An unbounded lattice is an infinite lattice. Natural number set is an unbounded lattice.
- For each element a, there exists an element $\neg a$ such that $a \veebar \neg a = 1$ and $a \barwedge \neg a = 0$.

Let X be a set. Then $(P(\mathcal{X}), \subseteq, \mathcal{X}, \phi)$ is a complemented distributive lattice i.e a Boolean Algebra where $P(\mathcal{X})$ is the power set of X.

Definition 36 *Let* $[\alpha] \leq [\beta]$ *iff* $\vdash_{A_x} \alpha \to \beta$

Definition 36 gives a partial order in $\mathcal{L}/\mathcal{R}$.

Theorem 5 *glb* $([\alpha], [\beta]) = [\alpha \barwedge \beta]$

Sketch of proof: First, we need to see whether $\alpha \wedge \beta \rightarrow \alpha$ is a tautology or not. From Table 7.1, it is seen that $\alpha \wedge \beta \rightarrow \alpha$ is a tautology. Similarly, we can find out that $\alpha \wedge \beta \rightarrow \beta$ is also a tautology. Hence, because of completeness of propositional logic, $\alpha \wedge \beta \rightarrow \alpha$ and $\alpha \wedge \beta \rightarrow \beta$ are theorems. So, according to the definition, $[\alpha \wedge \beta]$ is a lower bound of $[\alpha]$ and $[\beta]$. Let $[\delta]$ be also a lower bound of $([\alpha], [\beta])$ i.e. $\vdash_{A_x} \delta \rightarrow \alpha$ and $\vdash_{A_x} \delta \rightarrow \beta$ hold. Then it can be shown that $\vdash_{A_x} \delta \rightarrow (\alpha \barwedge \beta)$ holds. Hence, $\delta \leq \alpha \barwedge \beta$ and $[\delta] \leq [\alpha \wedge \beta]$. Therefore, $[\alpha \barwedge \beta]$ is the greatest lower bound.

Table 7.1: Truthtable of $\alpha \wedge \beta \rightarrow \alpha$

α	β	$\alpha \wedge \beta \rightarrow \alpha$
T	T	T
T	F	T
F	T	T
F	F	T

Similarly we can show that $[\alpha]$ and $[\beta]$ has the least upper bound $[\alpha \vee \beta]$.

Exercise: Show that,1. $\mathcal{L}/\mathcal{R}$ is a lattice. 2. $\mathcal{L}/\mathcal{R}$ is also distributive.

Theorem 6 *$\mathcal{L}/\mathcal{R}$ is complemented.*

Sketch of proof: $[\alpha] \veebar [\neg\alpha] = [\alpha \vee \neg\alpha] =$ The class of theorems. Again $[\alpha] \barwedge \neg[\alpha] = [\alpha \wedge \neg\alpha] =$ The class of contradictions. Now we shall see if the class of theorems is the top element and the class of contradictions is the bottom element or not. So, we need to show, any class $[\alpha] \leq$ the class of theorems and the class of contradictions $\leq [\alpha]$.

Verify that, all theorems are in one class. Let us take the class of theorems $[\gamma]$ (i.e. γ is a theorem). We need to show that $\vdash_{A_x} \alpha \rightarrow \gamma$ holds where γ is a theorem and α is any formula.

We can prove it by the axiomatic definition of the relation $\vdash_{A_x}$. Let us take a chain of formulae.

- γ [Since γ is a theorem]
- $\gamma \rightarrow (\alpha \rightarrow \gamma)$ [Axiom]
- $\alpha \rightarrow \gamma$ (M.P of the above two)

$\alpha \rightarrow \gamma$ is a theorem as $any\ formula \rightarrow theorem$ is a theorem.

Therefore, $[\alpha] \leq$ the class of all theorems. Similarly, we can show that the class of contradictions $\leq [\alpha]$.

Remark 1 *So, we can write the Boolean Algebra as,* $(\mathcal{L}/\mathcal{R}, \leq, \bar{\wedge}, \underline{\vee}, \neg\!\!\!\neg, 1, 0)$ *where 1 stands for the set of all theorems and 0 for the set of all contradictions (or anti-theorems). This is known as* ***Lindenbaum-Tarskey algebra of the logic****.*

Exercise: Complete the proof by filling in the gaps.

Chapter 8

Propositional Logic: Proof of Completeness Theorem

Class 9: Dated 08 **- November -** 2016

8.1 Completeness Theorem of Propositional Logic:

Statement: If $\Gamma \vDash \alpha$ then $\Gamma \vdash_{A_x} \alpha$. $\cdots$ (1)

8.1.1 Need for Completeness Theorem:

Say, we have a machine which takes a finite set of formula (Γ) and a formula (α) as input. The machine has information regarding the set of axioms A_x and the rule MP. The query is, whether it can decide $\Gamma \vdash \alpha$. See Figure 8.1.

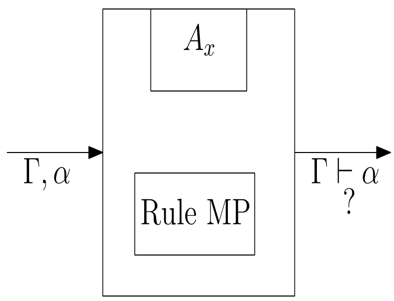


Figure 8.1: The hypothetical machine for completeness theorem

In general, this can not be decided. However, if along with these inputs, a derivation

$$\boxed{\begin{array}{c} \alpha_1 \\ \\ \alpha_2 \\ \\ \vdots \\ \\ \alpha_n \equiv \alpha \end{array}}$$

is also given and the machine is to check whether it is a derivation of α from Γ or not, then, the machine can check it. In other words, *"whether a sequence is a valid derivation or not"* can be checked by a computer.

Let us assume that, $\Gamma \vdash \alpha$ is true [Fact] from semantic angle. The question is, whether it is possible to get a sequence which proves its validity by axiomatic definition? In other words, whatever way we consider this to be a valid argument, is it also possible to get it syntactically? That means, assume for all valuations v,

$$v(\gamma) = T, for\ all\ \gamma \in \Gamma\ implies\ v(\alpha) = T.$$

That is, semantically α is a correct conclusion. Then, is it possible to obtain α from Γ? If α is obtainable from Γ from the axiom system, then the logical system is *complete*.

Example 23 In Eucledian axiommatic system, is it possible to derive the 5^{th} postulate from the first four postulates? That is,

$$Postulates\ 1 - 4 \vdash_{A_x} Postulate\ 5?$$

The statement is called *completeness*. The scenario is shown in Figure 8.2. The completeness theorem says that, this is obtainable.

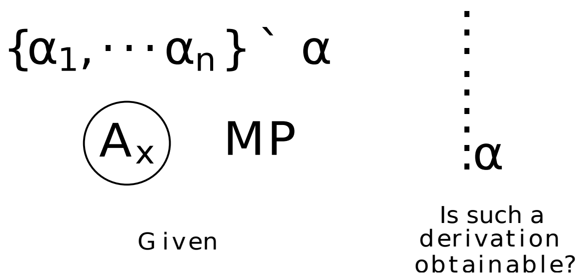


Figure 8.2: The completeness theorem

8.2 Proof of Completeness Theorem

The following proof is due to Leon Henkin. The proof is done in three steps:

Step 1: Any *consistent set* Γ of wffs can be extended to a *maximal consistent set* Δ [Lindenbanm Lemma].

Step 2: Δ has a model, that is, there is a valuation v, such that, $v(\Delta) = \{1\}$, i.e, *true*.

Step 3: The main proof of completeness statement.

First two steps are Lemmas and need to be proved before going to Step 3. Step 2 of the proof is the most important one. But before we go into the proof, let us define the following:

Definition 37 *A set Γ is called* inconsistent*, if and only if, there is a well formed formula α, such that $\Gamma \vdash_{A_x} \alpha$ and $\Gamma \vdash_{A_x} \neg\alpha$.*

That means, if a wff α is part of a consistent set, $\neg\alpha$ can not be derived from the set. However, each consistent set can be extended to a *maximal consistent set*.

Definition 38 *A set Δ of wffs is maximal consistent if and only if*

1. *Δ is consistent.*
2. *for any $\alpha \notin \Delta$, $\Delta \cup \{\alpha\}$ is inconsistent.*

That is, any formula added to it from outside, makes it inconsistent. See Figure 8.3.

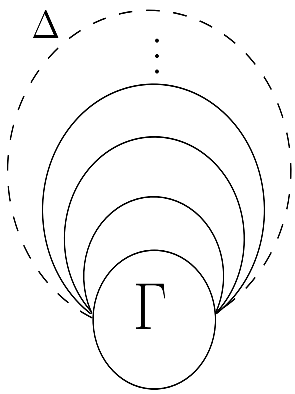


Figure 8.3: A maximal consistent set of Γ

Now, we can move on to the proof of completeness theorem.

8.3 Proof of Step 1 (Lindenbanm Lemma):

Proposition 9 *Any* consistent set Γ *of wffs can be extended to a* maximal consistent set Δ.

Proof: Let us enumerate the wffs of $\mathcal{L}$ as

$$\alpha_1, \alpha_2, \alpha_3, \cdots$$

Note that, although the alphabet is finite, but the formula set is infinite. The symbol $\cdots$ is used to represent the infinite continuity.

Now construct the sets of wffs

$$\Gamma_0, \Gamma_1, \Gamma_2, \cdots$$

as follows:

$$\begin{aligned}
\Gamma_0 &= \Gamma \\
\Gamma_1 &= \Gamma_0 \cup \{\alpha_1\}, && \text{if this is consistent} \\
&= \Gamma_0, && \text{otherwise} \\
\Gamma_2 &= \Gamma_1 \cup \{\alpha_2\}, && \text{if this is consistent} \\
&= \Gamma_1, && \text{otherwise} \\
&. \\
&. \\
&.
\end{aligned}$$

From this construction, it is obvious that,

$$\Gamma_0 \subseteq \Gamma_1 \subseteq \Gamma_2 \subseteq \cdots$$

Now take

$$\bigcup_i \Gamma_i = \Delta$$

[Mathematically, there is no problem as it is an well defined set, but computationally it is problematic.] The set $\bigcup_i \Gamma_i$ is defined as:

Definition 39 $\alpha \in \bigcup_i \Gamma_i$ *if and only if* $\alpha \in \Gamma_i$ *for some* i.

This also means that, for any wff α, either $\alpha \in \bigcup_i \Gamma_i$ or $\neg\alpha \in \bigcup_i \Gamma_i$

$\Delta = \bigcup_i \Gamma_i$ is the <u>maximal consistent set</u>.

8.3.1 Sketch of Proof:

Proof for Consistency of Δ: Let Δ be inconsistent, then there exists some β, such that $\Delta \vdash_{A_x} \beta$ and $\Delta \vdash_{A_x} \neg\beta$. This implies there exist two derivations

$$\beta_1, \beta_2, \cdots \beta_k (= \beta) \tag{8.1}$$

$$\beta'_1, \beta'_2, \cdots \beta'_m (= \neg\beta) \tag{8.2}$$

All the formulas of derivation 8.1 and derivation 8.2 must be from either $\Delta = \bigcup_i \Gamma_i$, or from Axioms or by MP rule. As these derivations are finite, they must have been included in some Γ_k which may be sufficiently large. Let us assume that, derivation 8.1 is from set Γ_k and derivation 8.2 is from set $\Gamma_{k'}$. As both $\Gamma_k, \Gamma_{k'} \subseteq \Delta$, that means according to the formation procedure either $\Gamma_k \subseteq \Gamma_{k'}$ or $\Gamma_{k'} \subseteq \Gamma_k$. In each case, the larger set derives both β and $\neg\beta$, which makes the set inconsistent. However, as per the formation procedure, all the Γ_is are consistent, so this is not possible. Therefore, Δ can not be inconsistent, because, if Δ is inconsistent, one of Γ_is must be inconsistent, which is not true. [Consistency of Δ is proved]

Proof for Maximality of Δ: Let Δ be not maximally consistent. Then there exists $\alpha \notin \Delta$ and a new set is formed as $\Delta \cup \{\alpha\}$ and $\Delta \cup \{\alpha\}$ is consistent.

Recall that, in the process of constructing monotonically increasing consistent sets, the wffs are arranged and enumerated as

$$\alpha_1, \alpha_2, \cdots, \alpha_i (= \alpha), \cdots$$

Therefore, if $\alpha_i = \alpha$, then the set $\Gamma_{i-1} \cup \{\alpha_i\}$ is already checked for consistency. If this set is found to be inconsistent, then in the procedure, $\alpha_i (= \alpha)$ is left out and not an element of the set $\Gamma_i = \Gamma_{i-1}$. As $\Gamma_i \subseteq \Delta$, so, $\alpha_i (= \alpha)$ can not be in Δ. In other words, if $\Gamma_i = \Gamma_{i-1} \cup \{\alpha_i\}$ is inconsistent, then as $\Gamma_i \subseteq \Delta \subseteq \Delta \cup \{\alpha\}$, so, Δ is inconsistent, which is not true. Therefore, $\Delta \cup \{\alpha\}$ is also inconsistent and α can not be added in to the set Δ.

On the other hand, if $\Gamma_{i-1} \cup \{\alpha_i\}$ is consistent, then $\alpha_i = \alpha \in \Gamma_i$. That means, $\alpha \in \Delta$.

So, Δ is the maximally consistent set.

There can be more than one ways of arrangements of the wffs α_is. So, there can be several maximally consistent sets (see Figure 8.4).

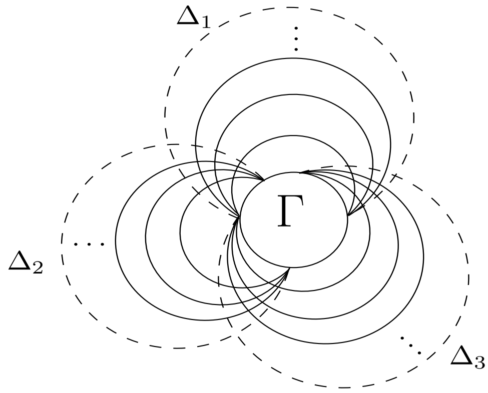


Figure 8.4: Some maximal consistent sets of Γ

8.3.2 Why is it necessary to prove consistency of Δ when it is true that each Γ_i is consistent?

We have taken $\Delta = \bigcup_i \Gamma_i$, that is, as arbitrary union over i of Γ_i. However, it can not be surely said whether the properties of individual consistent sets shall also hold over arbitrary union, because such union is only describable, we do not 'see' it. This is intuitively true, so to show it in any other way than intuition, we have to prove. There are many examples where the properties are finitely true, but not true over arbitrarily infinite set.

Example 24 Figure 8.5 gives an example of arbitrary union and intersection over two sets of points. In this figure, A and B are two sets of points on a line. Take $A \cap B$, which is non-zero length of interval and also a set of points. However, if we continue with this, the resulting set over arbitrary intersection $\bigcap[[[\cdots]]]$ is 1 singleton point.

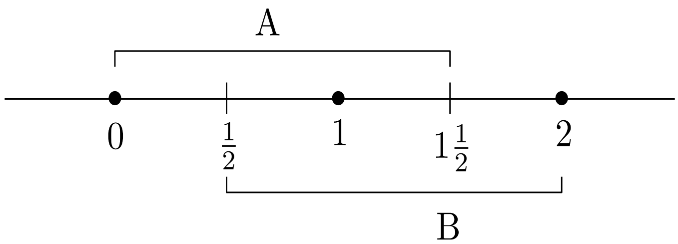


Figure 8.5: Arbitrary union and intersection over a set of points

8.3.3 Properties of Maximal Consistent Set

1. If $\Delta \vdash_{A_x} \alpha$, then $\alpha \in \Delta$. That is, nothing outside of Δ follows from Δ.

   Proof: Let $\alpha \notin \Delta$, then $\Delta \cup \{\alpha\}$ is inconsistent. That means, $\Delta \vdash_{A_x} \neg\alpha$. But as per the statement, $\Delta \vdash_{A_x} \alpha$. This violates consistency of Δ. So, $\alpha \in \Delta$.

2. $\alpha, \beta \in \Delta$ if and only if $\alpha \wedge \beta \in \Delta$.

3. $\alpha \vee \beta \in \Delta$, if and only if, $\alpha \in \Delta$, or $\beta \in \Delta$, or both.

4. For any wff α, either $\alpha \in \Delta$, or $\neg\alpha \in \Delta$ (see Figure 8.6).

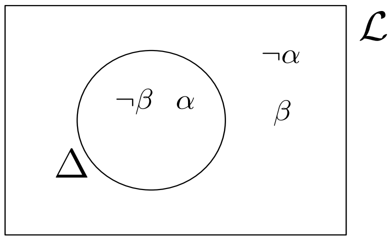


Figure 8.6: Language $\mathcal{L}$ and a maximal consistent set Δ

8.4 Proof of Step 2:

Let Δ be a maximal consistent set. We define a valuation v such that

$$\begin{aligned} v(p_i) &= 1 \;\text{ iff }\; p_i \in \Delta \\ &= 0 \;\text{ otherwise} \end{aligned}$$

Here, p_is are atomic wffs. Therefore , the question is what is the value for any formula $\alpha \in \Delta$. That is, $v(\alpha) =?$ for any $\alpha \in \Delta$.

8.4.1 To show that $v(\alpha) = 1$ iff $\alpha \in \Delta$

How to Prove: *Strong Induction* on complexity of the formula α.

Complexity 0: It is an atomic wff. So, directly follows from the definition.

Complexity n: Assumed that the statement holds for all wff with complexity $\leq n$

Complexity $n + 1$:

$\boxed{\textbf{Exercise: } \text{Complete the proof}}$

We now have a model, which sets any maximal consistent set to 1.

Therefore, from Step 1, any consistent set Γ has a maximal consistent extension Δ. By Step 2, Δ has a model, that is, $v(\Delta) = \{1\}$. So, as $\Gamma \subseteq \Delta$, $v(\Gamma) = \{1\}$. So, v is also a model of Γ. That means, any consistent set has a model.

8.5 Proof of Step 3:

Let $\Gamma \vdash \alpha$, but $\Gamma \nvdash_{A_x} \alpha$. Then $\Gamma \cup \{\neg\alpha\}$ is consistent, because, if $\Gamma \cup \{\neg\alpha\}$ is not consistent, then $\Gamma \vdash_{A_x} \alpha$.

So, by Step 2, there exists a valuation v, such that,

$$v(\Gamma \cup \{\neg\alpha\}) = \{1\}$$
$$\text{That is, } v(\Gamma) = \{1\} \text{ and } v(\neg\alpha) = 1$$
$$\text{That is, } v(\alpha) = 0$$

That means, v sets Γ to 1 and α to 0. This contradicts the assumption that $\Gamma \vdash \alpha$, that is, $v(\alpha) = 1$ if $v(\Gamma) = \{1\}$. Hence, completeness, i.e., if $\Gamma \vdash \alpha$, then $\Gamma \vdash_{A_x} \alpha$.

8.6 Issues in this style of proof:

Problems in this proof are:

1. Mathematically there is no problem; it is a nice proof.
2. Computationally there is some problem, because, firstly, the valuation function v is not constructive. Secondly, checking consistency is not simple.
3. This is a typical mathematical theorem. There is no clear method given to prove consistency of $\Gamma_n's$, as inconsistency may not reside in the surface.

There are some other points, like -

- In case of Modal logic, $\Gamma \vdash \alpha$ is complex and includes a relation R. So, to prove completeness, we need to change at Step 3.

  T Axiom: $\Box\alpha \rightarrow \alpha$. This means, if I know α, then α is true / α is a fact.

  [Reference for completeness proof for Modal logic: *Introduction to Modal Logic* by *Hudges and Cresswell*].

- There is an alternative proof: Kalmer's proof.
- There is no quantification in Propositional logic. But in case of Predicate logic, quantification (like $\forall, \exists$) exists, so, the prrof of completeness theorem changes.
- What is the use of non-computational constructive mathematics in society or real life or in computers?
  – We never use irrational number or recurring decimal number in computers, we take a rational approximation.

Chapter 9

First order predicate logic/First order logic (FOL)

Class 8: 15/11/2016

As before, we define logic as a pair $(\mathcal{L}, \vdash)$.

Alphabet

- $c_1, c_2, \cdots$ (constant symbols)
- $x_1, x_2, \cdots$ (variables)
- f_j^i where $i = 1, 2, \cdots$ and $j = 1, 2, \cdots$ (function symbols)
- p_j^i where $i = 1, 2, \cdots$ and $j = 1, 2, \cdots$ (predicate symbols)
- $\backsim, \rightarrow$ (logical connectives)
- $\forall$ (universal quantifier)
-),((left and right brackets)

Computers can not handle infinitely many symbols. So, to deal with this, use $c, x, |, \#, f, p, \backsim, \rightarrow, \forall,), ($. In the following way,

- $c_1 = c|, c_2 = c||, \cdots$
- $x_1 = x|, x_2 = x||, \cdots$
- $f_3^2 = f||\#\#\#$
- $p_3^2 = p||\#\#\#$ etc.

Other connectives and quantifiers are: $\wedge, \vee, \exists$. Here, $(\exists x \varnothing) \equiv \backsim \forall x$ $(\backsim \varnothing)$, $\vee$, $\wedge$ as in the case of propositional logic.

9.1 Definitions:

9.1.1 Terms:

1. Any c_i is a term.
2. Any x_i is a term.
3. If $t_1, \cdots, t_j$ are terms, then $f_j^i t_1 t_2 \cdots t_j$ is a term.
4. Nothing else is a term.

Example 25 $f_2^1 c_1 x_1$ is a term.
$f_2^2 f_2^1 c_1 x_1 c_2$ is a term. An example of such term is $(2+x) \times 3$, where $+, \times$ are two binary functions. It can be rewritten as $\times(+(2,x),3)$.

The subscript in the functions indicates the number of terms to follow. Writing a term in this way ensures *unique readability*, that is only one way of reading.

9.1.2 Well Formed Foumula (wff)

Atomic: If $t_1, \cdots, t_j$ are terms, then $p_j^i t_1 t_2 \cdots t_j$ is an atomic wff.

wffs: (1) Atomic wffs are wffs.
(2) If ϕ, φ are wffs, then $(\backsim \phi)$,$(\phi \rightarrow \varphi)$,$(\forall x_i \phi)$ are wffs.
(3) Nothing else is a wff.

Example 26 1. $p_2^1 c_1 c_2$ (example is $3 < 2$, written as $<(3,2)$)

2. $p_2^1 c_1 c_2 \rightarrow p_1^1 c_1$ (example is, $3 < 2 \rightarrow 3$ is prime).

   Predicate is a relation. Here $<$ is a two-place predicate (p_2^1), "*is prime*" is an 1 place predicate (p_1^1) and $3, 2$ are constants c_1, c_2 respectively.

3. Let us take the sentence, 'If $x > 10$ and $x < 15$ then x is good'.

   We can write this as $(p_2^1 x c_1 \wedge p_2^2 x_1 c_2) \rightarrow p_1^1 x_1$, where $p_2^1 x c_1$ represents the predicate $x > 10$, $p_2^2 x_1 c_2$ represents $x < 15$ and $p_1^1 x_1$ represents x *is good.*

Note:

- Superscript i on f_j^i and p_j^i is used to differentiate between the functions and the predicates, i.e enumeration of the j^{th} place functions or predicates.
- Predicates stand for mathematical relations (unary, binary etc).

- Functions are mathematical functions.
  - There is an ongoing point of debate in mathematics on the necessity of higher order language.
  - A relation is included in another relation, i.e. "$R_1(x, y)$ is a sub-relation of R_2" can be stated in FOL. But, according to a group of mathematicians, all situations, depicting about the relations/predicates, are not expressible in FOL. For example, we can not write a wff for $\forall f$ in FOL, but it can be written by second order logic.
  - Second order logic is used in mathematics.
  - However, some mathematicians say that higher order language is not essential. For example, we can replace $\forall f$ with natural language "for all f". Here using meta language, we extract the quantifier outside of the predicate.
- The set of wffs is decidable. Given a string whether it is wff or not can be checked by the machine.
- Axiomatic definition of turnstile (Hilbert type axiomatic system) is not the only way. However, in this lecture we shall adopt the axiomatic method.

Exercise: Take complicated sentences and symbolize them.

9.2 Free and Bound variables:

A position of a variable in a wff is free/bound. Note that, a wff is a string, which has finitely many positions (see Figure 9.1). In this figure, one variable x_2 has two positions in the string.

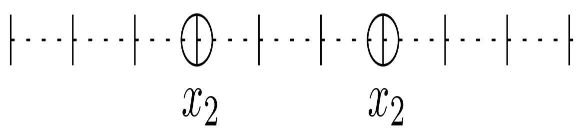


Figure 9.1: A wff (string) and its finitely many positions.

Bound:

1. The position just after a quantifier, e.g. - in $(\exists x\phi)$ or $(\forall x\phi)$, the variable x is bound by the quantifier $\exists$ or $\forall$.
2. The formula after x (bound) is called the scope of the quantifier. As an example, in $(\forall x\phi)$, x is bound by $\forall$ and ϕ is the scope of $\forall$.

3. Any position of x within the scope of the quantifier followed by x, is bound. In $(\forall x\phi)$, positions of x inside ϕ are bound.

Example 27 1. $\forall x(x + y = 5)$, where x is bound and y is free.

2. $\forall x \exists y(x + y = 5)$, where x and y both are bound.

3. $\forall x(x + y = 5) \to \exists y(y < 0)$ where, x is bound, y is free in $\forall x(x + y = 5)$ and y is bound in $\exists y(y < 0)$. So, for a variable in a formula, there can be both free and bound positions. See Figure 9.2.

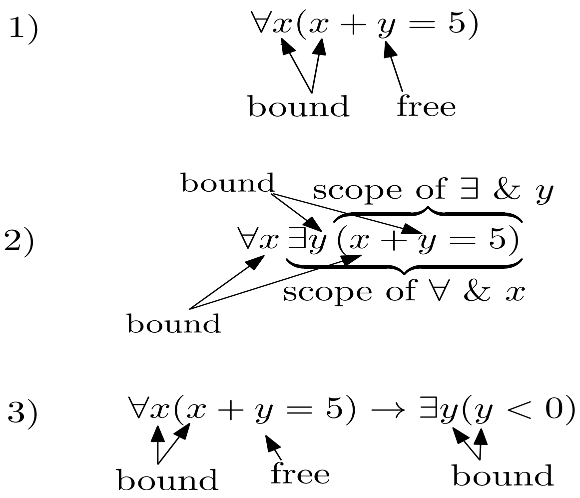


Figure 9.2: Examples of free and bound variables.

Definition 40 *For a variable x is said to be free/bound in a wff ϕ if and only if there are some free/bound occurrences of x in ϕ. Note that, a variable can be both free and bound at a formula.*

Notation: $\phi(t/x_i)$ stands for the wff obtained by replacing all the free positions of x_i in ϕ by the term t.

Example 28 Say, $\forall x(x + y = 5) \to \exists y(y < 0) \equiv \phi$. Then we can write $\forall x(x + (2 \times 7) = 5) \to \exists y(y < 0) \equiv \phi(2 \times 7/y)$. Here, y is replaced by $2 \times 7 \equiv f_2^1 c_1 c_2$ (in formal language).

Note that, more than one free variable can be replaced simultaneously, i.e. $\phi(t_1/x_1, t_2/x_2)$ as well as successively. For example, a wff $\phi(x, y, z)$ can be simultaneously replaced as $\phi(t_1/x, t_2/y, z)$ and successively replaced by first $\phi(t_1/x, y, z)$, then $\phi(t_1/x, y, z)(t_2/y)$.

A term is said to be closed if it has no variable. So, in case where term is not closed, resulting formula by simultaneous and successive replacement may not be same.

Definition 41 *ϕ admits t for x_i, if in $\phi(t/x_i)$ all the variables in the term t remains free.*

9.3 Derivation of FOL

9.3.1 Axioms of FOL:

1. $\phi \to (\psi \to \phi)$
2. $(\phi \to (\psi \to \varphi) \to ((\phi \to \psi) \to (\phi \to \varphi))$
3. $(\backsim \phi \to \backsim \psi) \to (\psi \to \phi)$
4. $\forall x_i(\phi \to \psi) \to (\forall x_i\phi \to \forall x_i\psi)$
5. $(\forall x_i\phi) \to \phi(t/x_i)$ provided ϕ admits t for x_i.

   eg: $\forall x \underbrace{(x < x+1)}_{\phi} \to \underbrace{2 < 2+1}_{\phi(2/x)}$, i.e, general to special/particular case.
6. $\phi \to (\forall x_i\phi)$ provided x_i is not free in ϕ. Here, x_i -free or not present makes this statement meaningless.
7. If ϕ is an axiom and x_i is free in ϕ, then $\forall x_i\phi$ is an axiom. This axiom replaces another rule: rule of generalization (cf. Mendelson).
8. Nothing else.

9.3.2 Rule:

MP: $\frac{\phi, \phi \to \psi}{\phi}$.

9.4 Define $\vdash_{A_x}$:

$\vdash$ is defined as in the case of propositional logic.

Definition 42 $\Gamma \vdash \phi$ *if there exists a chain $\phi_1, \phi_2, \cdots, \phi_n$ of wffs such that*

1. $\phi_n \equiv \psi$

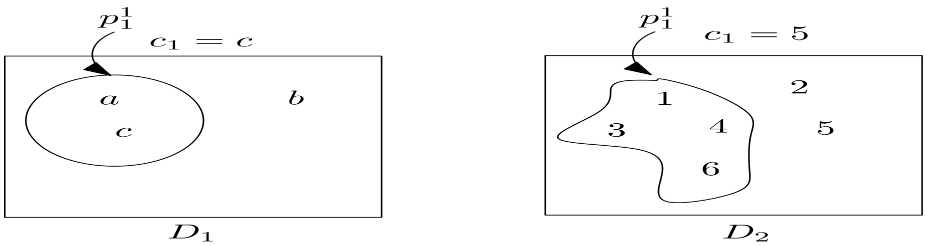
2. *any ϕ_i is either in the axioms, or in Γ or is obtained by the rule MP from the previous wffs of the sequence.*

Note:

- Constant, predicate, function symbols are called proper symbols, as these vary from one FOL to another.
- These are syntactically specific.
- Interpretation varies from domain to domain even in a particular FOL with one predicate symbol, one function and one constant.
- Without interpretation, we can not speak about truth or otherwise of a wff.

  e.g. Some interpretations of $p_1^1 c_1$: Say, in domain D_1, $c_1 = c \in \{a, c\}$ and the interpretation is shown in Figure 9.3a. Therefore, in this domain, $p_1^1 c_1$ is *true*.

  However, in domain D_2, $c_1 = 5 \notin \{1, 3, 4, 6\}$ (see Figure 9.3b). Therefore, in this domain, $p_1^1 c_1$ is *false*.



(a) (b)

Figure 9.3: Two interpretations of $p_1^1 c_1$

  So, here the interpretations of the one place predicate are the subsets of D_1 and D_2.

- One place predicate is subset of the domain and two place predicate is interpreted a subset of the cross product i.e. a binary relation.
- This interpretation is due to Tarski.
- The correspondence theory of truth is in syntactic form. Let us take an example, "Snow is white", i.e snow may not be ice, may be a name of a cow. Therefore, interpretation depends on domain and same wff can have different interpretation and truth values in different domains. See Figure 9.4.

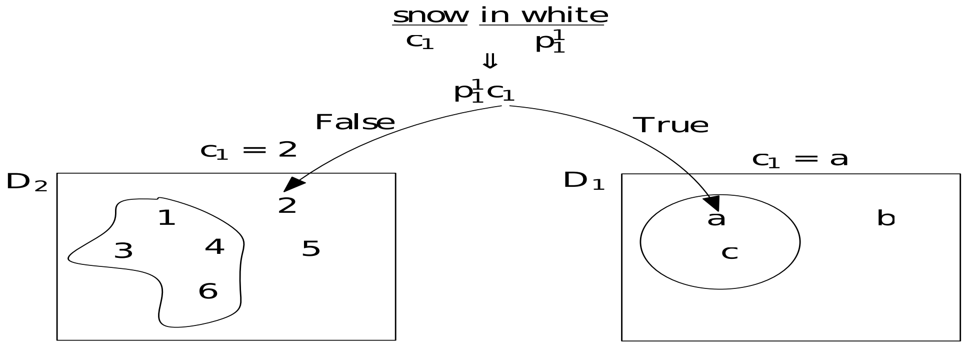


Figure 9.4: Interpretations of *Snow is white.*

In D_2, there are 6 numerical values that represent 6 objects. Among these objects 1, 3, 4 and 6 are white. They are kept in one domain. On the other hand, a and c are white objects and are kept in another domain (as shown in Fig. 9.4).

- There are always Infinite number of variables in a FOL.
- We can substitute the name of an object, not the object itself. So, interpretation is not on the variable, interpretation is on the meaning of the object.
- How to interpret a quantifier?
- All computer languages are context free. FOL is a context free language.

  1. Grammar can be said as Formation Grammar , i.e. formation of the language $\mathcal{L}$
  2. The formation of language and the derivation of one wff from another together form syntax.

  Task: Find a context free grammar that generates the FOL L (Formation Grammar).

- Compilers decide whether a string is in L or not.

Chapter 10

Predicate Logic: Satisfiability

Class 12: Dated 25 **- November -** 2016

Example 29 Is $x + 2 < 5$ a sentence?
The answer is 'No'. because –
(1) It has a free variable x; and hence
(2) It does not have a truth value, ie. truth value (T/F) cannot be assigned to it.

Now, to make it a statement, we have the following two options:

1. Replace the free variable x with an instance, like 4, that is, $4 + 2 < 5$ is a statement with truth value "False".
2. Place a quantifier before the free variable to make it bound. Like
   $\forall x(x + 2 < 5)$ with truth value "False".
   $\exists x(x + 2 < 5)$ with truth value "True".

However, $x+2 < 5$ as well as the other three are wffs according to the formation rules.

This implies that all wffs are not sentences. Variables are like *pronoun* of natural language. Eg. - "He is good". Here, as long as we don't know who is 'He', it is not a sentence with the truth value T/F. In ontology, the analogy is like putting a handkerchief on a seat and nothing on a seat. Handkerchief acts as a placeholder. So, in natural language using pronoun is like putting a placeholder; "— is good", where '—' is the placeholder.

However, we can not find the truth value of a wff without a domain. For example, for the wff ${p^1}_2{f^1}_2c_1c_2c_3$, the truth value can be assigned, when the domain is the set of numbers and the interpretation of the wff is $4 + 2 < 5$. Here, $c_1 = 4, c_2 = 2, c_3 = 5$, ${f^1}_2$ is the two place function $+$ and ${p^1}_2$ is the two-place predicate $<$.

We want to attach truth value to this formalization irrespective of interpretation. The wffs get values true or false or neither truth nor false through some domain.

10.1 Interpretation:

Interpretation can only be of a language. Here, the language is First Order Language of the logic FOL $(L, \vdash)$. By an interpretation I of the FOL language L, we mean the following:

- There is a non-empty set D, called the domain of interpretation.
- Each constant symbol c is associated with an element

$$I(c) \in D, \text{where } c \text{ is some constant symbol}$$

- Each function symbol ${f^i}_j$ is associated with a function

$$I({f^i}_j) : D^j \to D$$

- Each predicate symbol ${p^i}_j$ is associated with a j-ary relation

$$I({p^i}_j) \subseteq D^j$$

10.2 Notion of Satisfiability:

- Let I be an interpretation in the domain D.
- By s, we mean a sequence $\langle d_1, d_2, d_3, \cdots \rangle$ on D.

  This means, variable x_1 gets d_1, variable x_2 gets d_2 etc., $d_1, d_2, d_3, \cdots$ are taken from D.
- Given a term t, $s(t)$ is an element of D defined as follows:
  - If t is a constant c, then $s(t) = I(c)$. [That is, the meaning of such a t is independent of s and depends on I only.]
  - If t is a variable, say x_n, then $s(t) = d_n$. [That is, the sequence s is the assignment to the variables and x_n is assigned the n^{th} element of s.]
  - If t is ${f^i}_j t_1 t_2 \cdots t_j$, then

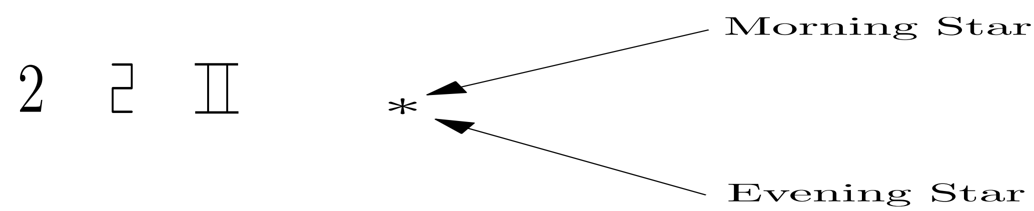
$$s(t) = I({f^i}_j)(s(t_1), s(t_2) \cdots s(t_j))$$

Example 30 $f^1{}_2 x_1 c_1$ is to be interpreted. Say, the domain is the set of numbers and $I(c_1) = 2$, $I(f^1{}_2) = +$. Then,

$$\begin{aligned} s(f^1{}_2 x_1 c_1) &= I(f^1{}_2)(s(x_1), s(c_1)) \\ &= +(d_1, I(c_1)) \\ &= +(d_1, 2) \end{aligned}$$

Here, $s(x_1)$ depends on the sequence s and $s(x_1)$ is the first element of s. Considering the first element of s in domain D as d_1, we get $+(d_1, 2)$ as an object of domain D. Then, we get the interpretation as $d_1 + 2$.

<u>**Note:**</u> c_1 and 2 are both symbols. Here, it is interpreted as if 2 is a real object. This is a kind of deception. It is a point of debate in Mathematics. In Mathematics, we can not escape the *names* and reach the real objects. For example, in Figure 10.1a, all the symbols are used to represent the physical quantity 'two'. But, all these are names, not the actual object. Similarly, in Figure 10.1b, the same astronomical object – planet 'Venus' is nicknamed as both *morning star* and *evening star*. That is, same object has different names, and all are again, symbols.



(a) Symbols representing 'two' **(b)** Nicknames for the planet *Venus*

Figure 10.1: Names vs Objects in Mathematics

Formal language of Mathematics can be translated to many, different natural languages. But, these will all be different symbols used in different languages. So, whatever be the language, these are naming of the objects, not the real objects.

10.2.1 s satisfies a wff ϕ:

Satisfiability is a binary relation. For a sequence s and a wff ϕ, the following cases can happen:

1. If ϕ is atomic, say $p^i{}_j t_1 t_2 \cdots t_j$, then, s *sat* ϕ iff

$$(s(t_1), s(t_2), \cdots s(t_j)) \in I(p^i{}_j)$$

Here, $(s(t_1), s(t_2), \cdots s(t_j))$ is a j-tuple of domain $I({p^i}_j)$ and $I({p^i}_j) \subseteq D^j$.

Example 31 Take a sentence $x + 2 < 5$, where $x = 3$. If $((3 + 2), 5) \in <$, then $x + 2 < 5$ is satisfied by $x = 3$. Formally, $x + 2 < 5$ is the wff ${p^1}_2{f^1}_2xc_1c_2$, where $I({p^1}_2) = <$. The satisfiability (SAT) problem is to determine whether a wff is satisfiable on some interpretation or not. In this particular case we need to determine if there is an interpretation I and a sequence s such that, $(s({f^1}_2xc_1), s(c_2)) \in I({p^1}_2)$. Let $s(c_2) = I(c_2) = 5$, $s(x) = 3$ and $s(c_1) = 2$. Then, $s({f^1}_2xc_1) = +(3, 2)$. So, $(+(3, 2), 5) \notin <$ for $x = 3$. That is, it is not satisfied for $x = 3$ and satisfied for $x = 1, 2$. That means, $x + 2 < 5$ is satisfied for some values of x and does not satisfy for some values of x. So, $p^1_2f^1_2xc_1c_2$ is staisfiable.

<u>**Note:**</u>

- Although s is infinite, we only need finitely many of them to give values to the variables in a string which is finite.
- If the domain is infinite, there are infinitely many ways to give values to the variables.
- Though we cannot talk about truth values of a formula with a free variable, we can talk about its satisfiability. For example, if $\phi = x+2 < 5$, then $\phi(3/x)$ is false. So the formula ϕ is not satisfied by a sequence that assigns 3 to x. But, $\phi(3/x)$ it is a different formula than ϕ.
- We cannot go to truth without passing through the notion of satisfiability.

2. s *sat* $\sim \phi$ iff s does not *sat* ϕ.

   s *sat* $\phi \rightarrow \psi$ iff either s does not *sat* ϕ, or, s *sat* ψ.

3. s *sat* $\forall x_i \phi$ iff $s(d/i)$ *sat* ϕ for all $d \in D$.

   [This is universal quantification on x_i, so only x_i is changed, other elements of the sequence remain unchanged.]

   Example 32 $\forall x_1(x_1 + x_2 < 5)$.

   Take $s = \langle 1, 3, \cdots \rangle$, where $x_1 = 1$, $x_2 = 3$ and so on. So, s *sat* $\forall x_1(x_1+x_2 < 5)$ implies, x_2 is fixed with the value 3 and on x_1, all the possible values are to be assigned from the domain. However, in sequence s, x_1 is also fixed. So, we need scope to change x_1 in all the elements of the domain. This notion is $s(d/1)$, meaning "d replaces 1^{st} element of s", where, $d \in D$.

   So, for all d, generate a set of sequences,

$$\boxed{\begin{array}{c} s(d/1) \\ \vdots \\ \\ \end{array}}$$

If $s = \langle 1, 3, 2, 2, \cdots \rangle$, $s(d/1) = \langle d, 3, 2, 2, \cdots \rangle$ and so on.

In this way, universal quantification is applied. If we have to evaluate $\forall x_i \forall x_j \phi$, we need to apply one quantifier at a time.

4. s *sat* $\exists x_i \phi$ iff $s(d/i)$ *sat* ϕ, for some $d \in D$.

Definition 43 *A wff ϕ is true with respect to an interpretation I iff s sat ϕ for all s over the domain of interpretation.*

Definition 44 *A wff ϕ is false with respect to an interpretation I iff s does not sat ϕ for any s over the domain of interpretation.*

If a wff is satisfiable for some assignments and not satisfiable for some other assignments, then it is neither true nor false. This theory of satisfiability is the contribution of A. Tarski. This theory is with respect to the *Notion of Truth* in Formal languages. The statement $x + 2 < 5$ is neither true nor false.

Note: This problem of satisfiability is of real importance. In predicate logic, it is needed to check all possible domains. So, it is in general undecidable.

Chapter 11

Satisfiability: continued

Class 14: Dated 2 **- December -** 2016

In the previous class we have learned,

1. ϕ is atomic formula say, $P^i_j t_1 t_2 \cdots t_j$, then s *sat* ϕ if and only if $(s(t_1), s(t_2), \cdots, s(t_j)) \in I(P^i_j)$.
2. s *sat* $\neg\phi$ if and only if s does not satisfy ϕ.
3. s *sat* $\phi \to \psi$ if and only if s does not satisfy ϕ or s *sat* ψ.
4. s *sat* $\forall x_i \phi$ if and only if $s(d/i)$ *sat* ϕ for all $d \in D$.

From the above four definitions, we can derive,

- s *sat* $(\phi \wedge \psi)$ if and only if s *sat* ϕ and s *sat* ψ
- s *sat* $(\phi \vee \psi)$ if and only if s *sat* ϕ or s *sat* ψ
- s *sat* $\exists x_i\ \phi$ if and only if $s(d/i)$ *sat* ϕ for some $d \in D$
- ϕ is true w.r.t I if and only if s *sat* ϕ for all s
- ϕ is false w.r.t I if and only if s does not satisfy ϕ for all s

Definition 45 *A wff ϕ is closed iff it has no free variable.*

Example: For example, say ϕ: $x < 2$. There is a free variable. Hence not closed. If $\phi(3/x)$: $3 < 2$, then it is closed as it has no free variable. Some other example are, $\forall x(x < 2)$, $\exists x(x < 2)$ - all these are closed and these are either true or false.

Proposition: A closed wff is either true or false with respect to any interpretation.

Either all the sequences satisfy the wff or all the sequences does not satisfy if the wff is closed. So, the wff is either true or false.

$x < 2$ is not a statement in mathematics.

We have always learned from school text books that, $(x+y)^2 = x^2 + 2xy + y^2$ where x, y are variables, '$+$' is the function symbol, '$=$' is the predicate and 2 is the constant. But the equation is actually a closed wff and it should be written as, $\forall x\ \forall y\ ((x+y)^2 = x^2 + 2xy + y^2)$.

Definition 46 $\Gamma \vdash \phi$, *iff for all interpretation* I*, if every member* γ *of* Γ *is true, then* ϕ *is also true. (Semantic Consequence relation).*

- "$\Gamma \vdash \phi$ if and only if $\Gamma \vdash_{A_x} \phi$" - does it hold? – This is actually completeness issue of predicate logic.

11.1 Categorical Proposition

Universal:

1. All men are mortal (universal affirmation): $\forall x\ (M(x) \rightarrow M_0(x))$
2. No man is mortal (univesal negation): $\forall x\ (M(x) \rightarrow \neg M_0(x))$

Particular:

3. Some men are mortal (Particular Affirmation): $\exists x(M(x) \wedge M_0(x))$
4. Some men are not mortal (Particular negation): $\exists x(M(x) \wedge \neg M_0(x))$

According to Aristotle, these are the basic propositions by which we can formalize any natural language.

Following is the derivation of universal negation.

$\forall x\ (M(x) \rightarrow \neg M_0(x)) \equiv \forall x\ (\neg M(x) \vee \neg M_0(x)) \equiv \forall x\ \neg(M(x) \wedge M_0(x)) \equiv \neg\exists x\ \neg\neg(M(x) \wedge M_0(x)) \equiv \neg\exists x\ (M(x) \wedge M_0(x)) \equiv$ direct negation of **'Some men are mortal'**.

Again, the particular negation can be derived by the following.

$\exists x(M(x) \wedge \neg M_0(x)) \equiv \exists\neg x(M(x) \rightarrow \neg\neg M_0(x)) \equiv \exists x\neg(M(x) \rightarrow M_0(x)) \equiv \neg\forall x\neg\neg(M(x) \rightarrow M_0(x)) \equiv \neg\forall x(M(x) \rightarrow M_0(x)) \equiv$ direct negation of **'All men are mortal'**

If there exists some sentences which are derivable within a theory and where one is not a contradiction to another, then the fact is known as consistency. Non-euclidean geometry is consistent i.e. non-contradictory.

Now the question is whether number theory is consistent or not. Or we can ask whether the system of natural numbers $\mathbb{N}$ is consistent or not. How do we decide that actually we shall never get a ϕ: ϕ is a theorem and $\neg\phi$ is a theorem.

11.2 *Gödel's Theorem*

If $\mathbb{N}$ is consistent, then $\mathbb{N}$ can not prove it.

This is the second incompleteness theorem. '$\mathbb{N}$ is consistent' has the representation in the language of $\mathbb{N}$ i.e. there is a wff which, if interpreted, results in '$\mathbb{N}$ is consistent'.

If we extend $\mathbb{N}$ to $\mathbb{N}'$ and $\mathbb{N}'$ can prove $\mathbb{N}$'s consistency, then there will arise a question whether $\mathbb{N}'$ is consistent or not. *Gödel's* second incompleteness theorem will then be applicable to $\mathbb{N}'$ and it will continue to $\mathbb{N}''$, $\mathbb{N}'''$, $\cdots$.

11.3 Euler's representation of statements in Set theory

It is the representation of statements what brain wants to see cognitively.

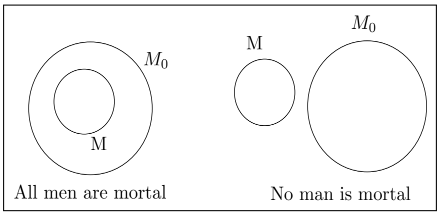


Figure 11.1: Euler Representation of universal proposition

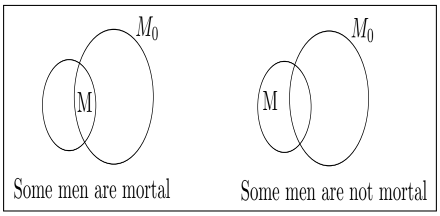


Figure 11.2: Euler Representation of particular proposition

Here, M is the representation for 'Men' and M_0 is the representation for 'Mortal'.

In Euler's representation, there lies some problems. If the pictorial representation be like Figure 11.3, then what should we understand by this representations? Because of the problem, Venn modified the representation.

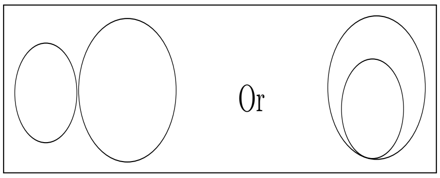


Figure 11.3: Problem in Euler Representation

11.4 Venn Diagram

The Venn diagram for the Universal propositions are given in Figure 11.4. The lines shades inside the circle represents the empty area.

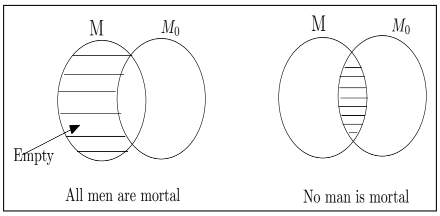


Figure 11.4: Venn diagram for universal proposition

Venn represented it in such a way that each curve would be divided into two parts by another curve. Figure 11.5 shows how each of the three curves are divided into two parts by every other curve.

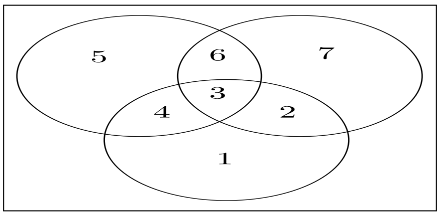
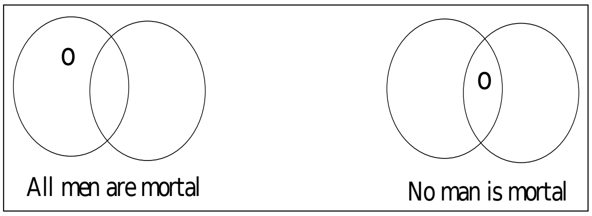


Figure 11.5: curves of venn diagram representation

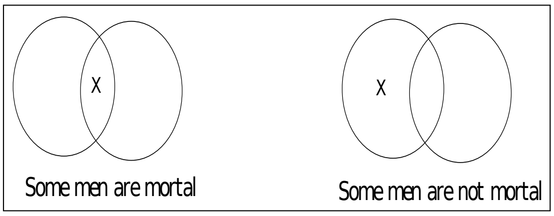
After Venn, Boole algebrized the propositional logic. Scientist philosopher Pierce marked the empty and non-empty zone of the curves more clearly in his representation.

11.5 Pierce Diagram

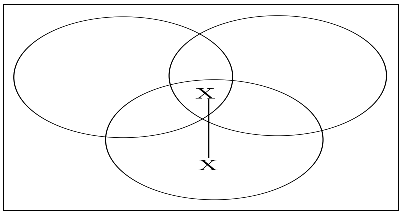
Pierce marked the empty zone by '0' and non-empty zone as 'x'. Figure 11.6a and Figure 11.6b show the Pierce representation for universal propositions and particular propositions respectively.



(a) Pierce Representation of universal proposition



(b) Pierce Representation of particular proposition



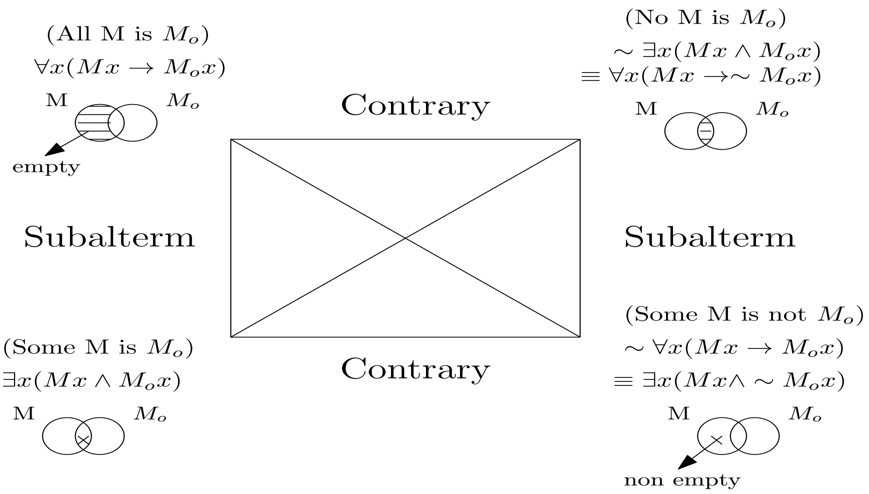
(c) Pierce Representation of disjunction

Pierce also tried to represent disjunction pictorially. Figure 11.6c shows the area ‘x–x’ that represents either a part including one x is non-empty or the part including another x is non-empty.

Chapter 12

First Order Logic

12.1 Square of Opposition:



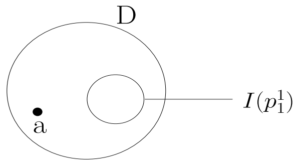
- ***Contrary:*** They can not be true together, but may be false together.
- ***Subcontrary:*** They can not be false together but may be true together.
- ***Subaltern:*** If the above corner is true then the corresponding lower corner is also true.

12.2 FOL:

- *Sentences* (closed wff) are either true or false.

- A *wff* is neither true nor false implies that it has a free variable but converse is not true.

  Example 33 $p_1^1 x_1 \rightarrow p_1^1 x_1$, Here x_1 is free variable, but this formula is always true.

  In this assignment, $p_1^1 x_1$ is not satisfied, so, $p_1^1 x_1 \rightarrow p_1^1 x_1$ is satisfied.

  In either case, this formula is satisfied. So, it is a valid wff and for all sequences it is satisfied, so it true.

  Example 34 $p_1^1 x_1 \wedge (\sim p_1^1 x_1)$

  For all assignments, this wff is false although x_1 is free here.

  That means, even if free variable is there a wff can be true or false.

- $\Gamma \vdash \phi$ holds if and only if for all interpretations I, if all the wff in Γ are true, then ϕ is also true.

  That means, if all wff in Γ are satisfied for all the sequences, then ϕ is also satisfied for all the sequences.

- $\Gamma \vdash' \phi$

  If all the wff in Γ are satisfied by an assignment s, then ϕ is also *sat* by s.

  Note:

  - Count only the sequence which satisfies all wff of Γ.
  - Consequence with definition more relaxed.

    If $\Gamma \vdash' \phi$ *then* $\Gamma \vdash \phi$, *converse does not hold*

  - If a sequence satisfies Γ then it satisfies ϕ. And if all sequence are true all sequence are satisfied. So, $\Gamma \vdash \phi$.

■ ***Question:*** $\Gamma \vdash_{Ax} \phi$ iff $\Gamma \vdash \phi$?

Answer: This holds, in case of sentences.

If $\Gamma \vdash_{Ax} \phi$ then $\Gamma \vdash \phi$ for all wff ϕ. The converse can be obtained only for sentences. The soundness holds always, but completeness holds when Γ, ϕ are closed wff.

$\Gamma \vdash_{Ax} \phi$ iff $\Gamma \vdash' \phi$ holds without restriction.

12.2.1 Soundness:

If $\Gamma \vdash_{Ax} \phi$ then $\Gamma \vdash \phi$.

Proof: Consider

1. All the axioms are true in any interpretation.
2. M.P preserves truth. i.e, if ϕ, $\phi \to \psi$ are true in some I then ψ is true in I.

Let $\Gamma \vdash_{Ax} \phi$, that is,

$$\begin{array}{c} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_n (\equiv \phi) \end{array}$$

Let in I, all members of Γ be true, we want to see, ϕ is true in I.

Now, if the elements of derivation α_1 is an axiom or a member of Γ or obtained by MP. That implies, if it is an axiom, by (i) it is true in all interpretations, so in I. Otherwise, it is true in I. Similarly α_2 is true in I. From α_3 onwards, by using (ii), i.e MP, it is true in I. So, $\Gamma \vdash \phi$ (Soundness proved).

12.2.2 Completeness:

If $\Gamma \vdash \phi$ then $\Gamma \vdash_{Ax} \phi$ where $\Gamma \cup \phi$ contain only sentences.

The proof is omitted.

12.2.3 Variations of FOL

- *FOL with equality prediction:* $t_1 = t_2$ (where '=' is binary relation)

- *Instances of FOL:* – Number Theory (Chapter 13)
  – Group Theory

Chapter 13

Prerequisite of Number Theory

Class 14: Dated 20 **- December -** 2016

Without '=' (equality) mathematics is not possible except in one/two cases, e.g. theory of order relation (Fig. 13.2).

13.1 Foundation of Mathematics:

There are two basic things in Mathematics:

1. Set theory
2. Logic

An example of first order theory is shown below viz. a Group structure.

13.1.1 Group

A group is a non-empty set G with a binary operation (.) and a 0-ary operation (e) satisfying the following conditions (also known as Group axioms):

1. $\forall x \forall y \forall z((x.y).z = x.(y.z))$
2. $\forall x(x.e = x \wedge e.x = x)$
3. $\forall x \exists y(x.y = e \wedge y.x = e)$

Here, all these three wffs are closed wffs and the 0-ary operation (e) is the *identity* object. As, '=' is a binary operation, so, $x.e = e.x = x$ is not proper, even computer can not take it. It is actually $x.e = x \wedge e.x = x$. Similarly, for the 3^{rd} axiom also, writing from both side with $\wedge$ is necessary; otherwise, left and right identity have been different and defined differently.

So, in this first order logic, the following three proper symbols are required

i) Constant e

ii) Function symbol $(\cdot)$

iii) Predicate symbol $(=)$

Here, '$\cdot$' is a binary (two place) function f_2^1 and '$=$' is a binary predicate p_2^1.

Note:

- After '$\forall$' only variables can be placed, so, $\forall x$, $\forall y$, $\forall z$ means x, y, z are variables. Hence, there is no need to say $x, y, z \in G$ for finite as well as infinite case.
- For infinite case, we can not explicitly describe the wffs using every element/object, so, there is no other way to describe the wffs without using variables.
- The variables in the three axioms are bounded.
- If a binary function f is defined over a set G, that means, G is closed under the function $f : G \times G \to G$. Here, all the operations are defined over a set, so, by definition *closed* property is satisfied – there is no need to mention *closed* explicitly.

  However, in case of interpretation, we need an *actual* group G' which must be closed under the three axioms, that is, these wffs are to be true in G'.

13.1.2 Function Correspondence

To define a function, we use the equality operator $(=)$. Of course, functions are relations, a n-place function is a $(n+1)$-place relation, whereas, special kind of relations are functions. Relations are more fundamental.

For example, the wff $\forall x(x.e = x \wedge e.x = x)$ is actually $\forall x(\cdot(x, e, x))$, where $\cdot$ is a 3-place relation which relates (x, e) to x, or (x, e, x) taken in this order $\in \cdot$.

We always write $2+3 = 5$. But this expression actually stands for '$+(2,3)\to 5$' or '$(2,3,5)\in +$'. In the first expression, $\to$ stands for corresponds to. In the second expression, $+$ is considered as a 3-place relation.

- **The theory in which the wffs are written in terms of equation is known as equational theory.**

So, there is a language and three formulae or group axioms. These group axioms may have models. All of the models that have the group axioms or closed wffs

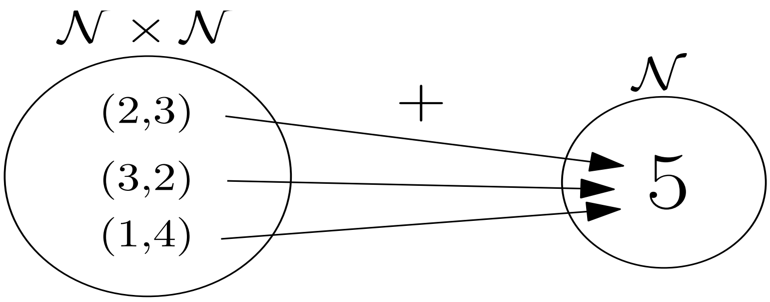


Figure 13.1: A binary function $+ : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is same as the ternary relation $+ \subseteq (\mathbb{N} \times \mathbb{N}) \times \mathbb{N}$

are called groups. These models are characterized by the axioms. Obviously, group axioms are not groups.

However, if there is no such model, then the axioms are inconsistent, but such axioms can be written. Note that, in the definition of group axioms, G is not mentioned. So, G is variable over the models or groups. In fact, G refers to any of these models.

13.1.3 Non-equational Theory

Example of non-equational wffs/axioms are:

- $\forall x(x < x) \Leftrightarrow (\forall x(p_2^1 xx))$
- $\forall x \forall y \forall z(x < y \wedge y < z \to x < z)$

where '$<$' is a binary predicate. Now, to get the model or interpretation characterized by these axioms, consider Figure 13.2. Here relation R is defined over set A which satisfies these axioms.

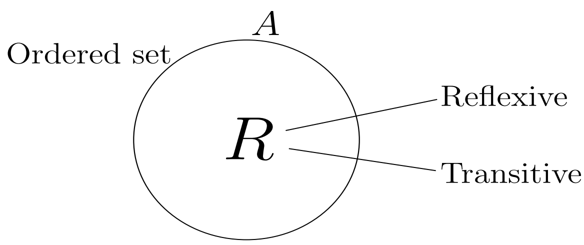


Figure 13.2: An ordered set with ordering relation R

Observe that, R is reflexive and transitive, so, the relation is called an ordering relation and the corresponding set is called an ordered set. The language for this model has only one binary predicate symbol, but no constant or equation. This is perhaps the only example of non-equational structure of wffs.

However, assuming general set as model means at least one set exists in which the properties hold; that is, the axioms are consistent. For instance, all problems given in any mathematics text book considers general set as model.

13.1.4 The notion of Semantic vs Syntactic

Now, consider Γ be a set of wffs in a first order language. Suppose that, Γ does not have a model (semantic). Then, Γ is inconsistent, i.e. there is a wff α such that $\Gamma \vdash \alpha$ and $\Gamma \vdash \neg\alpha$ (syntactic). To prove the syntactic part, we do not need any mathematical model. It can be derived logically from the axioms.

Theorem 7 : Γ *is consistent (syntactic) if and only if* Γ *has a model (semantic).*

This is also a statement of completeness theorem. When Γ is a closed wffs, then the statement is an essence of the first order language completeness theorem. Observe that, Γ is not specified in the theorem. But, we can say that, if we get a model, then the set Γ is always consistent.

Examples in any mathematics textbook are actually models. Hence, while defining a new definition/concept, one needs to give an example to prove its consistency. Therefore, definitions are abbreviations, while new definition needs a model.

We can now re-look at the basic concepts in Mathematics:

1. Axioms (Logic)
2. Model (Set)
3. Consistency proof (Logic)

13.2 First order language with equality (=):

Here, we are enhancing the first order language by incorporating a binary relation or two-place predicate symbol "=". So, in stead of the predicate p_2^k, '=' is to be used.

wffs:

• $= t_1 t_2$ where '=' is the predicate symbol and t_1, t_2 are terms. It is atomic, with no quantifier, conjunction or disjunction. By convention, we write it as $t_1 = t_2$.

• We need some additional wffs as axioms for equality.

(i) $\forall x(x = x)$ –(1)

(ii) $\forall x \forall y(x = y \rightarrow (\phi(x,x) \leftrightarrow \phi(x,y/x)))$ – (2)

If two elements are equal, then whatever can be said about the 1^{st} element can be said about the 2^{nd} element and vice-versa. In other words, if two elements are identical, then they have the same properties.

Note: $\phi(x,x)$ means wff ϕ where variable x can occur free in more than one places, whereas, $\phi(x,y/x)$ means the formula obtained where some of the occurrences of x are replaced by y.

Note that, "all properties of two elements are matched" implies, they are equivalent, but not necessarily same. So, how can we define the *identity/same*? If two objects have same properties, how can we know whether they are *same*? For this, we have the following principles given by Leibniz:

- $\forall x \forall y(x = y \rightarrow (\phi(x,x) \leftrightarrow \phi(x,y/x)))$ for all ϕ - this is known as *indiscernibility of identical.*
- $\forall x \forall y(x = y \leftarrow (\phi(x,x) \leftrightarrow \phi(x,y/x)))$ for all ϕ - this is known as, *identity of indiscernibles.*

But, is there any other way to infer sameness? By the two axioms ((1) and (2)) written above using predicate logic axioms, this can be proved –

Justification of equality:

$\forall x \forall y(x = y \rightarrow y = x)$ (commutativity)

$\forall x \forall y \forall z(x = y \wedge y = z \rightarrow x = z)$ (transitivity)

$\forall x \forall y(x = y \rightarrow t(x) = t(y/x))$ [e.g.$(x = y) \rightarrow (x^2 = y^2)$]

$\forall x \forall y(x = y \rightarrow (p_1^1 x \leftrightarrow p_1^1(y/x)))$ [e.g. $(x = y) \rightarrow (x.x = x.y)$]

where $t(x)$ is a term in which x occurs.

From Axiom (2) we can get:

$\forall x \forall y((x = y \wedge \phi(x,x)) \rightarrow \phi(x,y/x))$ [Using $\alpha \rightarrow (\beta \rightarrow \gamma) \leftrightarrow (\alpha \wedge \beta) \rightarrow \gamma$].

This means, if $x = y$ and x has the property ϕ, this implies, y also has the same property ϕ. This property is called *saturatedness property.*

Note:

- Number theory axioms are proper axioms (see next chapter).
- In the intuitive definition of set, the presumption is the notion of identity. For instance, if $a, b \in A$, where A is the set, $a \neq b$.
- However, in many occasions, having same property does not mean the objects will be equal:

- Although Morning star and Evening star refers to the same star,

$$Morning\ star \neq Evening\ star$$

- Suppose '$2 + 3 = 5$' is $fact(a)$ and '$4 + 1 = 5$' is $fact(b)$. Here, these are two different facts with different property. How shall you determine that they are same facts? In the first case, 3 is added to 2, and in the second case 1 is added to 4. So, intuitively they are not true, but, by the property of indiscernibles, mathematically they are equal.

- In Physics, there is no notation as one electron, where electron can be considered as model. The filed *Quantum Set Theory* deals with the objects that are distinct but not distinguishable.
- Although in mathematics there is no individualism, but in physics, we apply this theory to individuals. But some questions still remains:
  - Can a property of individual be really applied to the collection?
  - Can study of individual reflect collectivism? Or, we do it as we do not have any other way?

Chapter 14

Number Theory-I

Dated 30 **- December -** 2016

14.1 First Order Theory (FOT)

The base of FOT consists of

1. First Order Logic (FOL)
2. = (equality),

i.e. FOL with '=' and some closed wffs as proper axioms constitute FOT. Depending on the proper axioms, the theory changes. Constants, functions, predicate symbols are changed depending on the theory. One example of FOT is **Number Theory**.

14.2 Number Theory

In number theory, we need,

1. Proper symbols of the language:
   a) a constant c
   b) One 1-place function symbol and two 2-place function symbols: ${f_1}^1$, ${f_2}^1$, ${f_2}^2$
   c) equality '='

2. Other symbols:
   a) variables (enumerately many)
   b) $\neg$, $\rightarrow$, $\wedge$, $\vee$

c) $\forall$, $\exists$
d) (,)

Convention: a) instead of 'c' we write 'o' b) instead of ${f_1}^1$, ${f_2}^1$, ${f_2}^2$ we write $'$, +, ., c) for equality we write '='

Therefore, ${f_1}^1$c is equivalent to c'. Following are some examples of terms using the above conventions.
a) o
b) o', o'', ...
c) $o' + o''$ ...
d) $o'.o''$ ...
e) $(o' + o'').o'''$

Function ${f_1}^1$ is called successor function, ${f_2}^1$ is called addition function and ${f_2}^2$ is called multiplication function.

In pure Peano axioms, successor function and constant c is used. Currently we are not going to discuss pure Peano axiom. Instead we shall discuss about number theory with c, ${f_1}^1$, ${f_2}^1$, ${f_2}^2$.

14.3 Proper Axiom:

- N_1: $\forall x \forall y\ (x' = y' \rightarrow x = y)$:It is a closed wff. It means two different numbers can not have the same successor.
- N_2: $\forall x\ (x' \neq 0)$ (0 is not the successor of any x)
- N_3: $\forall x\ (x + 0 = x)$
- N_4: $\forall x \forall y\ (x + y' = (x + y)')$
- N_5: $\forall x\ (x.0 = 0)$
- N_6: $\forall x \forall y\ (x.y' = (x.y + x))$

if we want to write $m + n$, then the following scenario will appear. (Fig. 14.1)

And another axiom has to be mentioned in this context which is **Induction Axiom**.

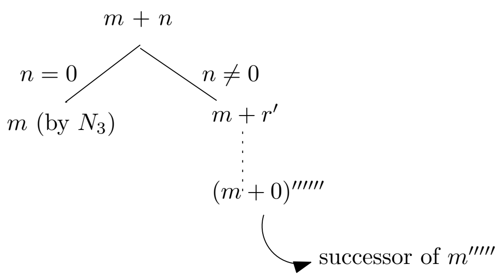


Figure 14.1: Understanding the successor

14.4 Induction Axiom:

- N_7: For any wff formula $\phi(x)$, the following wff is an axiom. $(\phi(0/x) \wedge \forall x(\phi(x) \rightarrow \phi(x'/x))) \rightarrow \forall x \phi(x)$ for any wff $\phi(x)$.

The induction axiom is a closed wff.
Standard understanding of induction: If 0 has a property ϕ and whenever x has a property ϕ and its successor x' has ϕ then every x has the property ϕ. This is the meaning of N_7.

Theorem 8 : $\forall x(x + 0' = x')$

We know what a successor is. But we don't know how to get that successor. It can be proved by the proof of the theorem.
Demonstration:

1. N_4: $\forall x \forall y\ (x + y' = (x + y)')$
2. $\forall x(x + 0' = (x + 0)')$ (Specialization rule (spec.))
3. $(x + 0' = (x + 0)')$ (Spec.)
4. $\forall x(x + 0 = x)$ (N_3)
5. $x + 0 = x$ (Spec.)
6. $x + 0' = x'$ (by Substitutivity, 5, 3)
7. $\forall x\ (x + 0' = x')$ (generalization)

In specialization, the quantifier is removed. In generalization the quantifier is added.

Chapter 15

Number Theory - II

Class 14: Dated 2 **- December -** 2016

Group Theory, Number Theory are the examples of FOT.

15.1 A Mathematical Theory (e.g. Group Theory)

1. A non-empty set plus some finite number of properties
2. Some operations and relations

Properties are the properties of operations and relations. For example, addition operation is the operation of Number Theory such as

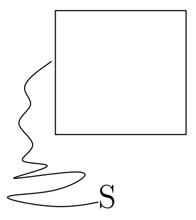
1. '+' is commutative
2. '+' is associative.

This is how by using symbolic and non-symbolic language, we can form rules. Mixture of symbols and natural language is a typical feature of scientific fields.

Usually properties are finitely many, whereas, instances of the properties are infinitely many.

According to Hilbert, we can build entire mathematics based on the basic axioms or some finite properties. Axioms are intuitive and their truth values are determined.

Finitistic Program



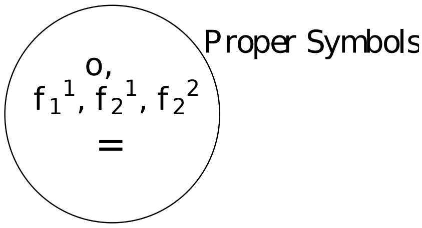
Suppose we have written a meaningful sentence S without knowing whether it is true or false. The finitistic program in the above figure contains finitely many information in the starting axioms and the methods are finitistic. Hilbert believed that there can be such a program from which any mathematical sentence S is either derivable or its negation is derivable.

Gödel said that whatever be the finitistic programme, an S can always be found where neither S nor $\neg S$ can be derivable from the finitistic program.

15.2 Formal Theory of Number Theory $\mathbb{N}$

FOL:

1. o
2. ${f_1}^1$, ${f_2}^1$, ${f_2}^2$
3. $=$ (predicate symbol)



It also includes variables
1. x_1, x_2, $\cdots$.
2. $\neg$, $\rightarrow$, $\wedge$, $\vee$
3. $\forall$, $\exists$, (,)
Atomic formulae with these symbols make non-atomic formulae.

15.3 Problem in Induction Axiom

For finite domain, we can verify the axiom. But for infinite domain we can not verify $\phi(x)$. Then what is the justification of $\phi(x)$ to be true?

Intuitionists say – give an algorithm; only then we can verify or establish $\phi(x)$. This is why computer scientists need an algorithm. Although constructivists do not believe in induction hypothesis.

Way to use induction axiom:

- First we need to derive $\phi(0/x)$.
- Then we prove $\forall x(\phi(x) \to \phi(x'/x))$
- Then $\forall x \phi(x)$ is obtained

15.4 Proof of $\forall x \forall y(x' + y = x + y')$

Say, $\phi(y) \equiv (x' + y = x + y')$ (We indicate the free variable y in this case)

Now we shall see whether we can establish $\phi(0/y)$. Next we shall see whether we can establish $\forall y(\phi(y) \to \phi(y'/y))$. If these two cases can be establish then by using induction axiom we will be able to say, $\forall y \phi(y)$ is true i.e. $\forall y \phi(y) \equiv \forall y(x' + y = x + y')$.

Therefore, we need to get $\phi(0/y)$. We will see whether $x' + 0 = x + 0'$.

1. $\forall x(x + 0 = x)$ (N_3)
2. $x' + 0 = x'$ (Spec.)
3. $x + 0 = x$ (Spec.)
4. $(x + 0)' = x'$ (Substitution rule)
5. $(x' + 0) = (x + 0)'$ (2, 4 transitivity of equality)
6. $x + 0' = (x + 0)'$ (N_4 Spec.)
7. $x' + 0 = x + 0'$ (5, 6 transitivity)
   Now we have to show, $\forall y(\phi(y) \to \phi(y'/y))$. (Deduction Theorem says, if from α we get β then we can say $\{\alpha\} \vdash \beta$ i.e. $\vdash \alpha \to \beta$). Here we derive $\phi(y'/y)$ from $\phi(y)$ from deduction theorem.
8. $\phi(y)$ is $x' + y = x + y'$ (Induction hypothesis or assumption)
9. $x' + y' = (x' + y)'$ (N_4 Spec.)
10. $x' + y' = (x + y')'$ (8, 9 Substitution)
11. $x + (y')' = (x + y')'$ (N_4 Spec.)
12. $x' + y' = x + (y')'$ (10, 11 transitivity)
13. $(x' + y = x + y') \to (x' + y' = x + (y')')$ (By deduction theorem)
14. i.e. $\phi(y) \to \phi(y'/y)$
15. $\forall y \phi(y) \to \phi(y'/y)$ (Generalization of predicate logic (Gen.))

16. $\forall y \phi(y)$ (induction axiom)

17. $\forall x \forall y \phi(y)$ (Gen.)

15.5 Internal and External Negation

$\nvdash 1 = 0$ and $\vdash 1 \neq 0$ are two different statements. When we write $\neg(1 = 0)$ then '$\neg$' is called internal negation and it is inside the language. But when we write '$\nvdash 1 = 0$', it is called external negation.

Let us give an example in natural language to understant this.
Example: A is <u>dis</u>honest. This is an internal negation.
If we say '<u>It is not that</u> A is honest then it is an external negation. The internal negation is stronger than the external negation. Therefore, internal negation is called strong negation and the external negation is called the weak negation.

15.6 Exercise

Prove that $\vdash 1 \neq 0$.

Chapter 16

Decidability

Dated 03 **- February -** 2017

16.1 Introduction

Statement: A set $\mathcal{A}$ is decidable if and only if there is an *effective procedure* for deciding whether or not any given object 'a' is or is not in $\mathcal{A}$.

However, this statement has the following problems:

1. Circularity: There is a word 'deciding' in the definition of *decidability*.
2. The words 'effective' and 'procedure' are not yet mathematically defined.

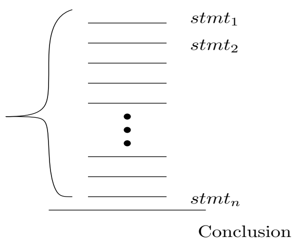


Figure 16.1: An effective procedure

In a layman's view, procedure means effective procedure. But, in mathematics, effective and non-effective procedures are different. In computer science, the term 'effective' is a formal mathematical property of procedures and an *effective procedure* is an *algorithm*. Mathematically, an effective procedure is a derivation having a set of mathematical statements (wffs), where each statement has either "Yes" or "No" as the truth values. In an effective procedure, every step is well-defined and follows from the set of Axioms or from the previous statements by using some rule (M.P.). See, for example, Figure 16.1.

In real life, almost all procedures are non-effective with fuzziness, ambiguity and jump in statement. For example, in medical science, a doctor's view is always subjective and does not follow the formal definition of effective procedure. In fact, almost 90% of the decisions taken by any human in his/her lifetime are non-effective with jump in between the arguments.

Remark: During this lecture, we have a doctor participant among us. His view on this context is, however, opposite of us.

16.2 Effective procedure and Turing Machine

To mathematically define decidability and effective procedure, we take help of Turing Machine. In fact, the famous work by Alan Turing and Alonzo Church on the solution of Hilbert's *EntscheidungsProblem* was on the precise meaning of *effective procedure*. This work is known as the *Church Turing Thesis* (1936).

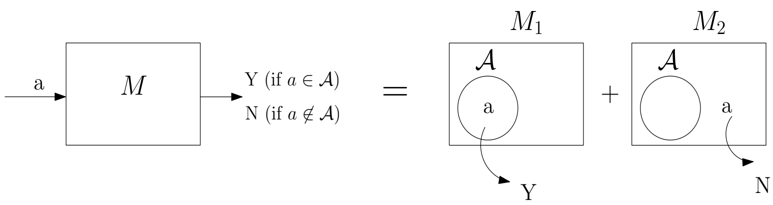


Figure 16.2: Effective procedure in terms of Turing Machine

Now, let us construct a machine M which integrates two machines M_1 and M_2. Here, the machine M_1 can decide whether $a \in \mathcal{A}$ and the other machine M_2 can decide whether $a \in \mathcal{A}^c$. In this case, the effective procedure is $M = M_1 + M_2$ (see Figure 16.2). Therefore, we redefine the notion of decidability as follows:

Definition 47 ***Decidable Set:*** *If a set $\mathcal{A}$ and its complement $\mathcal{A}^c$ can both be decided, then it is decidable.*

Definition 48 ***Enumerable Set:*** *A set is called enumerable if it satisfies the following properties:*

1. *It is an infinite set*
2. *The elements of the set can have a one-to-one correspondence with the elements of $\mathbb{N}$*
3. *The elements of the set can be written as $1, 2, 3, \cdots$*

A set is *countable*, if it is either finite or enumerable.

Definition 49 ***Recursively Enumerable:*** *A set $\mathcal{A}$ is said to be recursively enumerable if there exists a procedure that can pick up any element within $\mathcal{A}$.*

Definition 50 ***Recursive:*** *A set $\mathcal{A}$ is said to be recursive if there exists a procedure to determine whether $a \in \mathcal{A}$ and $a \in \mathcal{A}^c$.*

<u>**Note:**</u>

- The notion of decidable and recursive are practically same.
- A Turing machine can accept both the recursively enumerable and recursive languages. That is, we can build Turing machines for both the recursive and recursively enumerable languages. However, the general idea of algorithms refers to the recursive languages where the Turing machine halts for every input.
- Obviously, the set of recursive languages is a proper subset of the set of recursively enumerable languages.
- In terms of computability theory, a function is effectively calculable if and only if it is *partially recursive*. Whereas, a function is recursive if and only if it is effectively calculable and also a total function. Here, the notion of computability and decidability are the same.

To declare a set as recursive, we need to determine whether an element a belongs to the complement of the set. However, complement of a set is defined with respect to the *universal set*.

16.3 Universal Set

In classical set theory, the universal set is a closed set and predefined with respect to domain of inputs we are going to choose from. So, we can check whether an element belongs to a particular set of the closed universe or outside of that set in the universe.

However, the definition of closed universe in the classical set theory has long been a point of philosophical debate. The famous paradox proposed by Russell, also known as *Russell's Paradox* indicates this problem in the classical set theory.

To resolve this issue, formal set theory redefines the notion of universal set. It has been proved that "there is no universal set". So, to find the complement of a set, the concept of relative universe is used. In this case, the complement is defined relative to another set (see Figure 16.3 for example).

The model of the class of all (actual) sets are denoted by $\vee$ (see Figure 16.4).

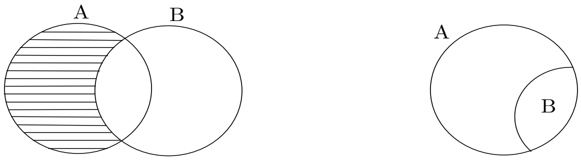


Figure 16.3: Relative universe and the set $\mathcal{A} \setminus \mathcal{B}$: objects of $\mathcal{A}$ which are not in $\mathcal{B}$

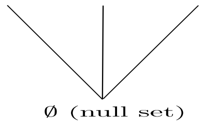
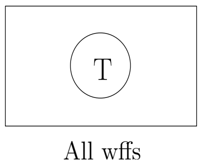


Figure 16.4: The model of the class of all sets

16.4 Decidability in Logic

In logic, a theory (T) is a set of closed wffs. Now, to be decidable, T should be subset of some set, which is the set of all wffs.



Definition 51 *A theory T is complete, if and only if, for any closed wff ϕ, either $T \vdash_{A_x} \phi$ or $T \vdash_{A_x} \neg\phi$.*

Recall that, in first order predicate logic A_x is the set of axioms and $T \vdash_{A_x} \phi$ means there exists a derivation like of Figure 16.5 which derives ϕ from the premise T using the set of axioms A_x.

However, not all theories are complete. In mathematics, *incomplete theories* do exist. If a theory T is complete, we have algorithm(s) to scan both inside and outside of T.

Theorem 9 : *If a theory T is complete, then it is decidable.*

Note: This statement is related to the complement of Gödel's incompleteness theorems. But, before going to the proof of this theorem, we need to have the following prerequisites.

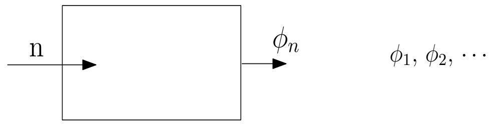
Theorem 10 : *The following statements are true for a decidable theory of logic:*

1. *The set $\{\phi \mid T \vdash_{A_x} \phi\}$ is effectively enumerable. That means, a machine exists which can give the n^{th} member of the set; or, in other words, an algorithm to decide the n^{th} member exists.*

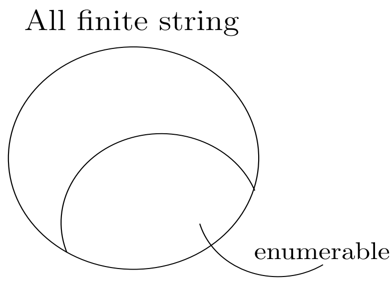
$$\left.\begin{array}{l}\alpha_1\\ \alpha_2\\ \bullet\\ \bullet\\ \bullet\end{array}\right\}\text{From } T\text{, or from } Ax \text{ or by MP}$$

$$\alpha_n \equiv \phi$$

Figure 16.5: $T\vdash_{A_x}\phi$ in FOL



2. *Further, this set* $\{\phi \mid T\vdash_{A_x}\phi\}$ *is technically infinite, because, if we can decide* ϕ*, then* $\neg\neg\phi$ *can be decided and so on. Here, each of* ϕ*,* $\neg\neg\phi$*,* $\neg\neg\neg\neg\phi$ *is a different string.*
3. *The set of alphabet is finite.*
4. *The set of all finite strings generated by this alphabet is enumerable.*



5. *Subset is either finite or enumerable.*
6. *We know* ϕ *is enumerable. Now, the theorems of* T *are effectively enumerable. Here, to be effective, we need to have an algorithm to define the enumeration of the theorems. A theorem can be written as the concatena-*

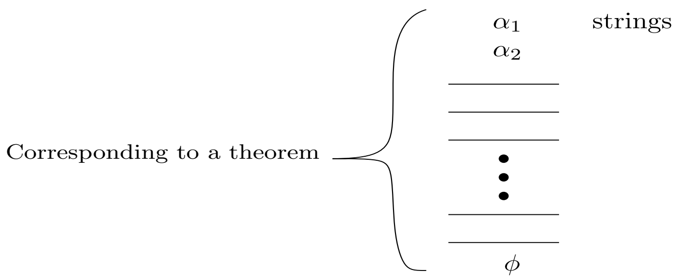


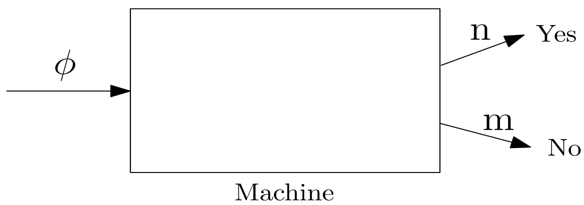
Figure 16.6: Derivation of a theorem

tion of the strings taken as steps of its derivation, like $\alpha_1\alpha_2\alpha_3\cdots$ *which is a finite string of finite strings. Hence, it is a number. So, whether something can be derived from* T *is to be tested by a machine and then a numbering* *is done.*

As an example, let us construct an algorithm which gives the n^{th} theorem of T. So, the input to the algorithm is n and T. So, the theory T can be defined as the set $T = \{\phi \mid T \vdash_{A_x} \phi\}$. This set is closed with respect to derivation.

We can define another set T^c. We need to show that T^c is also effectively enumerable.

Theorem 11 : *T is decidable, if T and T^c are both effectively enumerable.*



This implies, $T \vdash_{A_x} \phi$ is decidable, then there exists a machine which, given an enumeration n for which it is a theorem of T; or m, for which it is non-theorem of T^c. Now, we can prove Theorem 9:

Proof :

Case 1: Assume T to be consistent. Therefore, for any ϕ, either $T \vdash_{A_x} \phi$ or $T \vdash_{A_x} \neg\phi$ (from completeness).

In the first case, ϕ is located in the theory T by the previous result or observation. Whereas, in the second case, the procedure locates $\neg\phi$ in T. So, (the algorithm will decide that) ϕ is in T^c (from consistency).

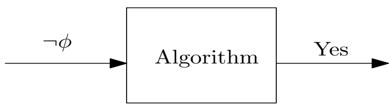


Figure 16.7: Algorithm to decide the T^c set

Note that, the completeness of T is the sufficient condition for decidability of T.

Case 2: Let T is inconsistent, then $\{\phi \mid T \vdash_{A_x} \phi\} = \mathcal{L}$.

From the classical set theory, if every set is a inconsistent, then every formula is derivable. So, the effective procedure has to find all elements within T, as T^c is empty. $\square$

Chapter 17

Gödel's Incompleteness Theorem

Class 18: Dated 10 **- February -** 2017

History:

1930: Ph.D thesis proving completeness of Russell's *Principia Mathematica* system. The completeness theorem he proved is:

If $\Gamma \vdash \alpha$, then $\Gamma \vdash_{A_x} \alpha$, where Γ is the Principia Mathematica system.

Special Case: Taking $\Gamma = \phi$, that is, no premises, only axioms in Γ, the completeness theorem is

$$\text{If } \vdash \alpha, \text{ then } \vdash_{A_x} \alpha$$

That is, if α is true (a tautology) in all interpretations (valuations), then α can be derived from the axioms. That is, if α is tautology from semantic point of view, then it is derivable from syntax. This is the proof of completeness.

1931: Incompleteness theorems. Kurt Gödel gave the idea about these theorems first in an informal meeting in a restaurant in Austria on the eve of a conference, where both he and David Hilbert were speakers. At the time of an informal meeting outside the conference-sessions, John von Neumann was present, and probably he gave Hilbert hints about Gödel's incompleteness theorems. Previously, Hilbert's claim was *"Anything speakable in Mathematics is either provable or disprovable"*. This idea was called *Hilbert's Finitistic Programme*. However, this programme fails, if Gödel's incompleteness theorems are true.

17.1 Formal Number Theory

Formal number theory N has 1 constant symbol 'o' and a 1-place function symbol '$'$'. So, the notations are $o', o'', \cdots$.

17.1.1 Symbolizing

Informally, we know o' as 1. We symbolize it as $\overline{1}$. So, the numerals of N are

$$
\begin{gathered}
o \\
o' : \overline{1} \\
o'' : \overline{2} \\
\vdots
\end{gathered}
$$

Let ϕ be a formula with free variables $x_1, x_2, \cdots x_n$, that is, $\phi(x_1, x_2, \cdots x_n)$. Substituting with numerals we get, $\phi(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$, which is another formula with no free variables.

17.1.2 Representing Predicates

In informal number theory/system, predicates are used for representing relations. Let $\mathscr{P}$ be an n-place relation of informal number theory, such as $\leq$ (a 2-place relation). However, in formal number theory, we do not have any predicate symbol except '$=$'. Therefore, to represent a predicate symbol or relation of informal number system, we need a formula ϕ with n number of free variables. Note that, any formula with a free variable is a property.

Definition 52 *The relation $\mathscr{P}$ of an informal number theory is said to be "expressible" in N, if and only if, there is a formula ϕ, such that – if $\mathscr{P}(a_1, a_2, \cdots a_n)$ holds, then $\vdash_N \phi(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$, otherwise, $\vdash\sim \phi(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$.*

In informal number system, $\mathscr{P}(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$ is either true or false. If it is true, then $\phi(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$ is derivable (or a theorem). If $\mathscr{P}(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$ does not hold, then negation of $\phi(\overline{a_1}, \overline{a_2}, \cdots \overline{a_n})$ is derivable, where $\overline{a_1}$ is the corresponding formal linguistic symbol of a_1. It can be noted that, some realtions of informal number system can not be *expressible* in formal number system.

Example 35 Take the binary relation '$\leq$'. So, we need a formula in N with two free variables. Let this formula be

$$\phi(x, y) \equiv \exists z(x + z = y)$$

As, z is bound here, so, ϕ is a formula having two free variables.

Claim: The predicate relation $\leq$ is expressible by $\phi(x, y)$. that is,

$$\vdash_N \exists z(\overline{2} + z = \overline{5}) \text{ [That is, this theorem is provable]}$$

$$\text{and } \vdash_N \sim \exists z(\overline{5} + z = \overline{2})$$

Example 36 Relation " is even". This is an one-place predicate and is a subset relation in informal number theory (see Figure 17.1).

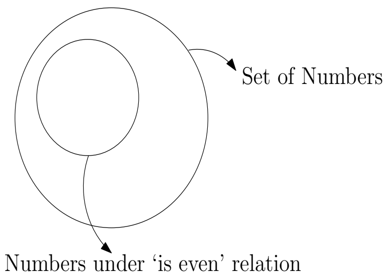


Figure 17.1: 'is even' relation in Number theory

We need a formula with 1 free variable, like

$$\exists z(x = \overline{2}.z)$$

Here $\overline{2}$ is the constant corresponding the informal number theory symbol 2 and z is bound in the formula; so, the free variable is x.

Note: In informal number system, a relation $R \subseteq \mathbb{N}^k$. So, for a 1-ary relation $R \subseteq \mathbb{N}$. So, 1-ary relation is subset relation. In mathematical logic, 0-ary relation depicts to the constants, that is the objects of the domain.

Peano's notion: Predicate symbol $=$, function symbols $+, .$ and constant 'o' are sufficient to express all relations of number theory.

17.2 Gödel Numbering:

This is a method that assigns an unique actual number (in informal number theory) to each symbol of the alphabet, each well formed formula and each sequence of well formed formulas.

Also, this scheme ensures that, given any number, it will be possible to decide–

1. whether it is a Gödel number or not,
2. if it is so, what is the corresponding syntactic entity [that is, alphabet symbol, wff or strings of symbols]

17.2.1 Outline of Method:

Step 1: • To each symbol, assign odd numbers (from the beginning), like - $1, 3, 5 \cdots$ etc.

[Note that, as number of symbols in alphabet is finite, we need only finitely many odd numbers from the beginning.] We denote, by $\mathtt{g}(\Box)$, the number corresponding to the symbol $\Box$.

- We need a sequencing of the symbols in the alphabet to do the numbering.

  For example, see Figure 17.2. Note that, the symbols x and $|$ can construct infinitely many variables.

Symbols :	$\sim$	$\rightarrow$	(	)	$+$	.	$=$	$'$	x	$\vert$	o	$\exists$
	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$	$\downarrow$
Numbering :	1	3	5	7	9	11	13	15	17	19	21	23

Figure 17.2: One arbitrary numbering of symbols in Number theory

Step 2: A formula is a finite sequence of symbols. To each formula $S = s_1 s_2 s_3 \cdots s_k$, where s_i is any symbol, assign

$$2^{\mathtt{g}(s_1)}.3^{\mathtt{g}(s_2)}.\cdots.p_k^{\mathtt{g}(s_k)}$$

where p_k is the k^{th} prime from the beginning. This is denoted by $\mathtt{g}(S)$.

Example 37 $S = \exists x_1(x_2 = \overline{2}.x_1)$. This formula can be rewritten as

$$\exists x|(x|| = o''.x|)$$

So, $\mathtt{g}(S) = 2^{23}.3^{17}.5^{19}.7^{5}.11^{17}.13^{19}.17^{19}.19^{13}.23^{21}.29^{15}.31^{15}.37^{11}.41^{17}.43^{19}.47^{7}$

Note: The concept of a number corresponding to a formula is so unique a phenomenon that, nothing ever happened in the history parallel to this!

Step 3: To each sequence of wffs $S_1, S_2, \cdots, S_n$, assign the number

$$2^{\mathtt{g}(S_1)}.3^{\mathtt{g}(S_2)}.\cdots.p_n^{\mathtt{g}(S_n)}$$

Note: If two sequences are not same, then there exists at least one place, where the numbering will be different because of the different symbol used. So, it is not possible to assign two different formulas the same Gödel number.

17.2.2 Finding the syntactic entity of a Gödel number:

Before giving the procedure to find the syntactic entity corresponding to a Gödel number, one needs to consider as assumptions the following fundamental theorems:

Theorem 12 : *Any positive whole number n can be expressed/factorized as powers of primes. This is called the* ***fundamental theorem of Arithmetic****.*

Theorem 13 : *There are infinitely many primes. This is a* ***fundamental theorem of intuitive number theory****.*

Proof for Theorem 13 is given by Euclid. These theorems are used implicitly.

Note that, the Gödel number of formulae and sequences are even numbers, because, in each case, the first prime used is 2. So, given a number n, we need to check the following:

- If n is odd, check the (finitely many) symbols of alphabet. If no match is found, n is not a Gödel number.
- If n is even, check if the number can be factorized in terms of successive primes or not. If not, n is not a Gödel number.
  Otherwise, say, the number is factorized as $2^a.3^b.5^c.\cdots$
  - If either of $a, b, c, \cdots$ is even, then the numbering is not of a formula. If all are odd, check for the corresponding strings. However, even valid string in the above sense may not always give wffs. For example, $2^3.3^1.5^{13}$ corresponds to $\rightarrow\sim=$ (see Figure 17.2), which is not a wff. But, whether it is a wff or not, is decidable.
  - If a, b, c are all even, that is like $2^{2n_1}.3^{2n_2}.5^{2n_3}$, then, it can be the numbering of a sequence of sequences.

For example, 10 is not a Gödel number, because– (i) it is not odd, so not a symbol, (ii) $10 = 2^1.5^1$, and $2, 5$ are not successive primes.

17.2.3 Further Discussion:

Reason for Formalized Number System:

1. Even whole numbers of intuitive/informal number theory need to be formalized.
2. There also exists some theorems which need to be formalized, like a statement – *'There exists infinitely many primes'*.

Some other points:

1. N is the object language. Target is to translate a meta-linguistic statement into the statement of the object language, ie, a formula of N.
2. Any expressible relation of meta language is first translated into an equivalent relation of informal number theory; then it is translated into a formula of N (if it is expressible).

3. Greek *'Liar's Paradox'* is an example of self-reference. E.g. - "I lie". In some sense, this can be considered as a predecessor to Gödel's thought.

4. Let a formula have a proof. This means, a formula, which is a sequence of symbols, has a Gödel number (a) associated with it and similarly, a proof, which is a sequence of sequences, also has a Gödel number (b) for it. So, a number (b) is the proof of another number (a).

   Therefore, we can define a binary relation between numbers. This relation between numbers may be called *proof relation/derivability relation.*

5. Each proof has a unique Gödel number associated with it. For example,

$$\begin{array}{lll} & \alpha & \alpha' \\ & \alpha \to \beta & \alpha' \to \beta' \\ \hline \text{MP} & \beta & \beta' \end{array}$$

   the Gödel numbers for these two proofs are different if $\alpha, \alpha', \beta, \beta'$ are different.

6. A number n can be provable by another number m, if and only if, there exists a sequence of sequences $S_1, S_2, S_3, \cdots S_k$ which represents the Gödel number m and n represents S_k.

7. A number is called expressive, if there is a formula which represents n in N.

8. Gödel will translate the sentence 'N is inconsistent' within N by using Gödel numbers.

Chapter 18

Gödel's Incompleteness Theorem: continued

Class 18: Dated 02 **- March -** 2017

Recall, N is the formal system of natural numbers and $\mathbb{N}$ is the set of actual numbers. $1, 2, 3, \cdots \in \mathbb{N}$ and $o, \overline{1}, \overline{2}, \cdots$ are symbols belonging to N.

Definition 53 <u>*ω-consistency:*</u>
N is ω-consistent if and only if for <u>every</u> wff $\phi(x)$ with exactly one free variable x, $\vdash_N \phi(\overline{n})$ for <u>every</u> n implies $\nvdash_N \neg\forall x\phi(x)$ i.e. $\nvdash_N \exists x\neg\phi(x)$.

Here, by replacing x with the number $\overline{n}$, the formula is no longer with free variable. Like, $x < \overline{2}$ is converted to $\overline{1} < \overline{2}$, which is closed.

<u>Note:</u>

1. If we knew N to be consistent, then $\vdash_N \phi(\overline{n})$ for every $\overline{n}$, would have implied $\vdash_N \forall x\phi(x)$ and $\nvdash_N \neg\forall x\phi(x)$. But we do not know about the consistency of N, so, it is necessary to assume $\nvdash_N \neg\forall x\phi(x)$. [Recall, consistency: $\vdash_N \phi \Rightarrow \nvdash_N \neg\phi$, for all ϕ.]
2. $\vdash_N \phi$ means $N \vdash_{A_x} \phi$ (syntactic), where A_x is the set of axioms of formal number system N.

Theorem 14 : *[From classical logic:] $\Gamma \vdash_{A_x} \phi, \neg\phi$, if and only if for all well formed formula ψ, $\Gamma \vdash \psi$.*

The part '$\Gamma \vdash_{A_x} \phi, \neg\phi$' is called *negation inconsistency* and 'for all well formed formula ψ, $\Gamma \vdash \psi$' is called *explosiveness or absolute inconsistency*. This theorem states that, if Γ is inconsistent, then everything follows from Γ. This comes from the theorem,

$$\vdash_{A_x} \phi \to (\neg\phi \to \psi) \ .$$

In classical logic, negation inconsistency is equivalent to absolute inconsistency. So,

$$\boxed{\Gamma \text{ is consistent, if and only if, } \Gamma \nvdash \psi, \text{ for some } \psi.}$$

Lemma 1 *If N is ω-consistent, then N is consistent.*

Proof : Let N be ω-consistent. Take a well-formed formula $\phi(x)$ with 1 free variable x. Two possibilities are there –

1. $\vdash_N \phi(\overline{n})$, for all n.
2. $\nvdash_N \phi(\overline{n})$, for some n.

For Case 1, $\nvdash_N \neg\forall x\phi(x)$ because of ω-consistency; for Case 2, $\nvdash_N \phi(\overline{n})$. So, in either case, we have at least a formula which does not follow from N. Hence, N is consistent. □

Note: This style of proving consistency is from classical logic, which is against common sense. Alternative logic is *para-consistency* popularized by Brazilian mathematician *Newton da Costa*.

Also note that, if $P \to Q$, then it means, Q is more general, $P \subseteq Q$.

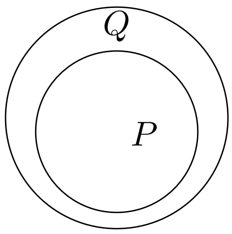


Figure 18.1: Sets of $P \to Q$

So, consistency is a more general notion than ω-consistency.

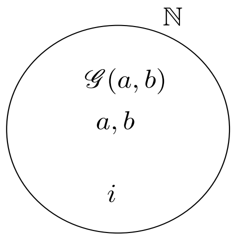
18.1 1^{st} Incompleteness Theorem:

Theorem 15 : *[1^{st} Incompleteness Theorem:] If N is ω-consistent, then there is a well formed formula S, such that $\nvdash_N S, \nvdash_N \neg S$.*

Hilbert's idea was that, given any mathematical formula, it is derivable or its negation is derivable.

Definition 54 *Let $\mathscr{G}$ be a 2-place binary relation in $\mathbb{N}$ defined as following:*

$\mathscr{G}(a, b)$ holds if and only if 'a' is the Gödel number of a wff $P(x)$ with exactly one free variable x and b is the Gödel number of one of the proofs in N of $P(\overline{a})$.



$P(\overline{a})$ must be a theorem with at least one proof in N. $P(\overline{a})$ is replacement of x with corresponding Gödel number of a, where a is the Gödel number of $P(x)$.

Claim: $\mathscr{G}$ is expressible in N. [No proof]

Expressibility is expressing a statement outside of a system by a formula inside that system. Example of expressibility: $x < y$ can be expressed as $\exists z(x + z = y, z \neq 0)$.

18.1.1 Main proof of 1^{st} incompleteness theorem

Let $\mathsf{G}(x, y)$ be the formula that expresses $\mathscr{G}$. Let us take the formula

$$\forall y \neg \mathsf{G}(x, y) \equiv P(x)$$

Let i be the Gödel number of $P(x)$, where $i \in \mathbb{N}$ and $\bar{i} \in N$. Take,

$$P(\bar{i}) \equiv \forall y \neg \mathsf{G}(\bar{i}, y)$$

We take $P(\bar{i})$ as S. We shall prove:

1. If N is consistent, then $\nvdash_N S$.
2. If N is ω-consistent, then $\nvdash_N \neg S$

- **Proof of (1):**

  Assumption, N is consistent. Now, if possible, let $\vdash_N S$, that is,

  $$\vdash_N \forall y \neg \mathsf{G}(\bar{i}, y) \qquad \cdots \qquad (A)$$

  So, it has proof(s). Take a proof of this wff. Let, j be the Gödel number of this proof. So, $(i, j) \in \mathscr{G}$. Since, $\mathscr{G}$ is expressible,

  $$\vdash_N G(\bar{i}, \bar{j}) \qquad \cdots \qquad (B)$$

  However, from (A), using rule *Spec.*, we get

  $$\vdash_N \neg G(\bar{i}, \bar{j}) \qquad \cdots \qquad (C)$$

This (C) along with (B) contradicts our assumption of consistency of N. So, $\nvdash_N S$.

Note: *If a property ϕ holds for all x i.e. $\forall x \phi x$, $\phi(t/x)$. So, $\forall x \phi x \to \phi(t/x)$ holds. This is the specialization rule in predicate logic.*

- **Proof of (2):**

  Assumption, N is ω-consistent. If possible, let $\vdash_N \neg S$, that is

$$\vdash_N \neg\forall y \neg \mathsf{G}(\bar{i}, y) \qquad \cdots \qquad (A')$$

Since, N is ω-consistent, so N is also consistent. So, $\nvdash_N S$. That means, no number n is there. which could be the Gödel number of a proof of S. So, $(G)(i, n)$ does not hold for n. Therefore, by expressibility

$$\vdash_N \neg G(\bar{i}, \overline{n})$$

Now, $\neg G(\bar{i}, y)$ is a wff with exactly one free variable y. Therefore, by ω-consistency,

$$\nvdash_N \neg\forall y \neg \mathsf{G}(\bar{i}, y) \qquad \cdots \qquad (B')$$

Note that, here (A') and (B') are meta-statements about two facts which contradicts. So, (A') and (B') cannot be together. Hence, $\nvdash_N \neg S$.

So, from 1 and 2, we can say that, if N is ω-consistent, then there is a wff S, such that, $\nvdash_N S$ and $\nvdash_N \neg S$.

18.2 2^{nd} Incompleteness Theorem:

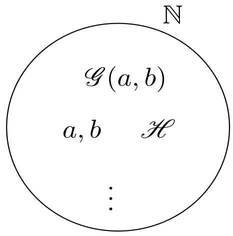
This is the main incompleteness theorem. This theorem is due to Gödel and Rosser.

Theorem 16 : *[Gödel and Rosser Theorem:] If N is consistent, then there is a well formed formula R, such that $\nvdash_N R$, $\nvdash_N \neg R$.*

18.2.1 The formula R:

Definition 55 *Define another binary relation $\mathscr{H}$ in $\mathbb{N}$ as following:*

$\mathscr{H}(a, b)$ holds if and only if,' a' is the Gödel number of a formula $P(x)$, 'b' is the Gödel number of a proof of $\neg P(\overline{a})$.



$\mathscr{H}$ is expressible in N, say by $H(x,y)$. Let a formula

$$\phi(x) \equiv \forall y(G(x,y) \rightarrow \exists z(z \leq y \wedge H(x,z))$$

Note that, x is the only free variable and both y and z are bound by the universal operators $\forall$ and $\exists$ respectively. Now, let j be the Gödel number of $\phi(x)$. Take,

$$\phi(\bar{j}) \equiv \forall y(G(\bar{j},y) \rightarrow \exists z(z \leq y \wedge H(\bar{j},z))$$

This is our R to prove this Theorem 16.

Note: Rosser sentence R roughly says that, if a statement is provable in N, then its negation is already provable. That is, if there is a proof of R with a Gödel number n, then there is a proof of $\neg R$ with another Gödel number.

18.3 3^{rd} Incompleteness Theorem:

Informally, this theorem states that, *if N is consistent, then this fact cannot be proved in N.*

Here, N is the formal system with formula, axioms etc. Now, the statement 'N is consistent' is a meta-statement, not a wff of N. So, we need to express it by some wff of N (see Figure 18.2). For this, we first define the following relation –

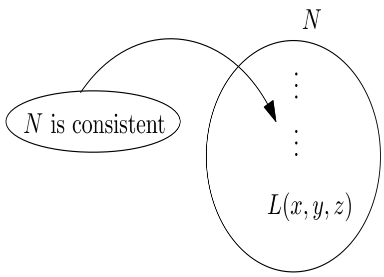
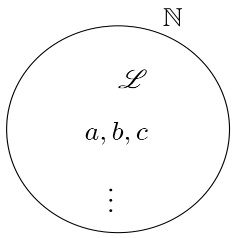


Figure 18.2: The problem of 3^{rd} Incompleteness Theorem

Definition 56 *$\mathscr{L}$ is a 3-place relation defined in ℕ as follows:*

$\mathscr{L}(a, b, c)$ holds if and only if, 'a' is the Gödel number of a formula ϕ, 'b' is the Gödel number of a proof of ϕ and 'c' is the Gödel number of a proof of $\neg\phi$.

$\mathscr{L}$ is expressible in N by $L(x, y, z)$. Note that, defining a relation does not necessarily means that it has to hold.



Now, we can get a well formed formula

$$\neg\exists x \exists y \exists z L(x, y, z) \qquad \cdots \qquad CONS$$

which stands for the meta-statement '*N is consistent*'.

Note: The wff $\exists x \exists y \exists z L(x, y, z)$ means, there exists a formula ϕ, for which proofs of both ϕ and $\neg\phi$ are present in N; so, N is inconsistent. Therefore, $\neg\exists x \exists y \exists z L(x, y, z)$ means, N is consistent.

Theorem 17 : *[Statement of 3^{rd} Incompleteness Theorem:]*
If N is consistent, then $\nvdash_N CONS$.

Symbolically, it is the last nail to Hilbert's program!

Note: There has been many researches on avoiding this incompleteness issue from N, such as –

1. Increasing or extending the axiom set to prove the completeness in the extended axiom set. But, it is found that, this concept will not work as long as expressibility of N is included in the system.

2. Selecting a small part of the axioms of N to avoid this scenario.

Chapter 19

Fuzzy Set Theory

May, 2018

The boundary of a set may not be sharp always. That is, it can not be always *crisp*. This is the main motivation of studying *Fuzzy set theory*. Fuzzy set theory deals with the scenarios where the boundary is not crisp.

For example, 'Tall men', 'Wise men' etc. In case of 'Tall men', one does not know *how much tall*. If you are given a number, by this number you can say whether someone is tall or not. However, the number is not fixed. In case of the example of *tall men*, atleast one gets a number. But when you are asked how much wise a man can be, you do not even have any numerical value. So you have to fix a range within which a person can be decided as tall or wise. That means, the boundary is not sharp. These are not sets or a proper collection of objects according to classical set theory.

According to classical set theory, set is a collection of *well defined* and *distinct objects*. In case of fuzzy set theory, it does not fulfill the characteristics: well defined and distinct. (Mainly *well defined*).

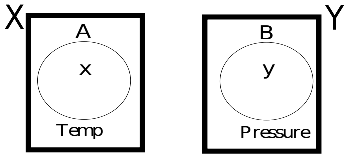
If there is a classical set A, then there is a characteristic function χ_A that corresponds to A.

Similarly, a fuzzy subset A in the universe X is defined by the function $\mu_A : X \to [0, 1]$ and $\mu_A(x)$ is called the degree of membership of x in the fuzzy set A. This idea can be extended to fuzzy r

elations.

19.1 Classical Conditional Proposition

I. If x is A then y is B. This conditional statement may be expressed by a relation between variables x and y.



$\chi_B(y) = \sup_{x \in X} \min(\chi_A(x), \chi_R(x,y))$
where, R is a relation $R \subseteq X \times Y$ and χ_A, χ_B are the characteristic functions.

Generally, one can define a composition $\circ$ between a set $A \subseteq X$ and a relation R. This composition in the classical set theory is defined as follows.
$A \circ R \subseteq Y$ such that $y \in A \circ R$ if and only if there exists $x \in A$ such that, xRy holds. In other words, $\{y|xRy\}$ is the image of A under R.

II. $\chi_R(x,y) = 1$ iff either $\chi_A(x) = 0$ or $\chi_B(y) = 1$. So, **I** may be expressed as the relation R in **II**. Thus, if you know A and R, you can obtain B. In other words, given $xinA$ and *if x is A then y is B*, i.e. (x,y) is R, we can determine B as $A \circ R$.

This rule is extended in the case of fuzzy logic where A, B are fuzzy subsets and R is a fuzzy relation. More specifically, Let us have the information,

If x is A
If x is A then y is B where A, B are fuzzy subsets
Then we can write,
x is A
(x,y) is R where $R(x,y) = A(x) \to B(y)$ for some implication operation.
Then we conclude,
y is $A \circ R$.
It will turn out that, $B = A \circ R$. Thus we get an alternative version of M.P. More generally we can have,

III.
If x is A then y is B
If x is A'
Then y is $B' \equiv A' \circ R$.

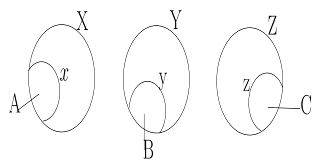
To note that R is the fuzzy implication relation defined by using the fuzzy sets A and B but B' is obtained by composing it with A'. **III** is called generalized M.P.

Note: Zadeh's logic is semantic, the sup-min composition is the basic method of calculation of the truth value. This is the universal core. However, the methodology is dependent on the value set [0, 1] and taking "min" for the conjunction operator. The composition has been generalized by Peter Hajek and many others.

This composition method may be applied to generalized hypothetical syllogism by the following schema:

If x is A then y is B
If y is B then z is C

If x is A then z is C



In other words,
If $< x, y >$ is $A \to B \equiv R_1 \subseteq A \times Y$ $\quad$ $A(x) \to B(y)$
If $< y, z >$ is $B \to C \equiv R_2 \subseteq B \times Z$ $\quad$ $B(y) \to B(z)$

---

then $< x, z >$ is $A \to C \equiv R_3 \subseteq A \times Y$ $\quad$ $A(x) \to B(y)$

Using the compositional method, one gets,
$R_3(x, z) = \sup_{y \in Y} \min(R_1(x, y), R_2(y, z))$
i.e. $R_3 = R_1 \circ R_2$.

It is surprising that, the relation $R_1 \circ R_2$ turns out to be $A \to C$
i.e. $R_1 \circ R_2(x, z) = A(x) \to C(z)$ for all x, z.
Thus we can see the semantic counterpart of the rule HS. Any rule that involves a sup-min composition is called a <u>Compositional rule of inference</u>. M.P and H.S are two such rules.

The compositional rule may be applied more generally.
Let $A_1, A_2, \cdots A_n$ be fuzzy subsets on $X_1, X_2, \cdots X_n$. Let there be a fuzzy subset P on $X_1 \times X_2 \times \cdots X_k$ i.e.
$P(x_1, x_2, \cdots, x_k) \in [0,1]$ i.e. a k-ary relation. Let R be a fuzzy relation on $X_1, X_2, \cdots X_n$. Then,
$P \circ R = \sup_{x_1, x_2, \cdots, x_k} \min[P(x_1, x_2, \cdots, x_k), R(x_1, x_2, \cdots, x_k, y_{k+1}, \cdots, y_n)]$. $P \circ R$ is a fuzzy relation on $X_{k+1} \times \cdots \times X_n$, where
$P \circ R(y_{k+1}, \cdots, y_n)$ is given by the above expression. The process is first extending then projecting:
For a fixed $y_{k+1}, \cdots, y_n$, we first extend by all choices of $(x_1, x_2, \cdots, x_k)$.
Obtain a value from sup-min. Then project that value on $(y_{k+1}, \cdots, y_n)$.

Class notes on 17/05/2018

19.2 Fuzzy Implication Operator:

In classical logic,
$\alpha \to \beta \equiv \neg\alpha \vee \beta$
$\equiv \neg\alpha \vee (\alpha \wedge \beta)$
$\equiv (\neg\alpha \wedge \neg\beta) \vee \beta$

If $v(\alpha) = a$, $v(\beta) = b$, where $a, b \in \{0, 1\}$
then, $v(\neg\alpha \vee \beta) \equiv \max((1 - v(\alpha)), v(\beta))$
Again, we can write, $v(\neg\alpha \vee (\alpha \wedge \beta)) \equiv \max((1 - v(\alpha)), \min(v(\alpha), v(\beta))$
These are equivalent in classical logic, but not in many valued logic. When $a, b \in [0, 1]$, then the values do not remain the same. That is, the functions are not identical when the value set is, [0, 1]. So we have different implication operators (refer to the book by Klir and Yuan, p. 309).

Let us define an implication:

$$a \to b = \sup_x \{x | a \wedge x \leq b\} \tag{19.1}$$

This implication is called *residuation*. If we keep the value set {0, 1}, then we get exactly the same truth table as that of the classical logic. In other words, we can write this implication by residuation. Eq. 19.1 is actually the deduction theorem. Deduction theorem is the syntactic way of logic, whereas, the semantic part is the residuation expression.

Deduction Theorem:
$\Gamma, \alpha \vdash \beta \iff \Gamma \vdash \alpha \to \beta$

$\gamma, \alpha \vdash \beta \iff \gamma \vdash \alpha \to \beta$ (special case)
This in turn implies that,
$\gamma \wedge \alpha \vdash \beta \iff \gamma \vdash \alpha \to \beta$

The corresponding algebraic expression to the above equation is,
$x \wedge a \leq b$ iff $x \leq a \to b$.

'$\leq$' corresponds to '$\models$' for fuzzy logic. In many valued logic, if α is designated by a value then β is also designated by a value. However, it is not necessary that '$v(\alpha) \leq v(\beta)$' for all valuations v. This is the difference between many valued logic and fuzzy logic.

> **Exercise:** Show the equivalence between,
>
> i) $a \to b = \sup\limits_{x}\{x | a \wedge x \leq b\}$ and ii) $x \wedge a \leq b$ iff $x \leq a \to b$

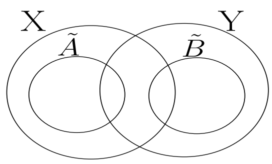
Now, let $\tilde{A}$ and $\tilde{B}$ be two fuzzy sets. If x is $\tilde{A}$ then y is $\tilde{B}$. The semantic translation is, $\tilde{R}(x, y) = \tilde{A}(x) \to \tilde{B}(y)$. The generalized M.P rule,

If x is $\tilde{A}$ then y is $\tilde{B}$
If x is $\tilde{A}'$

---

y is $\tilde{B}' = ?$

Corresponding $\tilde{R}$ is a fuzzy relation. Therefore, y is $\tilde{B}' = \tilde{A}' \circ \tilde{R} = \sup_x \min(\tilde{A}(x), \tilde{R}(x, y))$.



Modus Tollens: In classical logic,

If $\alpha \to \beta$
If $\neg\beta$

$\neg\alpha$

In Fuzzy logic

If x is $\tilde{A}$ then y is $\tilde{B}$ $\quad$ $\tilde{R}(x,y)$
y is not $\tilde{B}$ $\quad$ $1-\tilde{B}(y)$

---

x is $(1-\tilde{B})\circ\tilde{R}$ $\quad$ $\sup_y \min(1-\tilde{B}(y),\tilde{R}(x,y))$

(For Modus Tollens List, refer to the book by Klir and Yaun, P. 316)

Generalized Modus Tollens:
If x is $\tilde{A}$ then $y is \tilde{B}$
y is $\tilde{B'}$

---

x is $\tilde{B'}\circ\tilde{R}$

Generalized HS:
If x is $\tilde{A}$ then y is $\tilde{B}$ $\quad$ $\tilde{R_1}(x,y)\equiv\tilde{A}(x)\to\tilde{B}(y)$
If x is $\tilde{B'}$ then z is $\tilde{C}$ $\quad$ $\tilde{R_2}(y,z)\equiv\tilde{B'}(y)\to\tilde{c}(z)$

If x is $\tilde{A}$ then z is ?
z is $R_1\circ R_2$ i.e. $\sup_y \min(\tilde{R_1}(x,y),\tilde{R_2}(y,z))$. This is the relation between x and z.

Multiconditional Approximate Reasoning:

If x is $\tilde{A_1}$ then y is $\tilde{B_1}:\tilde{R_1}$
If x is $\tilde{A_2}$ then y is $\tilde{B_2}:\tilde{R_2}$
.
.
.
If x is $\tilde{A_n}$ then y is $\tilde{B_n}:\tilde{R_n}$
x is $\tilde{A}$

---

y is ? $(\tilde{A}\circ\tilde{R})$

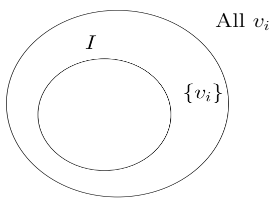
One of the methods to solve this is, $\tilde{R}=\tilde{R_1}\wedge\tilde{R_2}\wedge\cdots\wedge\tilde{R_n}$. Hence, $\tilde{R}(x,y)=\min(\tilde{R_1}(x,y),\tilde{R_2}(x,y),\cdots,\tilde{R_n}(x,y))$. This is one of the methods that gives good results sometimes. One application is washing machine where fuzzy logic controller is used.

19.3 Semantic Notion of Graded Consequence

We define in many valued context, $X \models \alpha$ (Semantic Consequence). α follows from X to some degree. We shall define this motion. We shall write,
$gr(X \models \alpha) \in [0,1]$. This is the generalization of classical semantic consequence relation. In classical case, $gr(X \models \alpha) \in \{0,1\}$ means, for all valuations v_i, if every member of X is 1, then α also gets 1 under v_i.
That is, If $X \subseteq v_i$, then $\alpha \in v_i$.
$X \models \alpha \equiv \forall v_i(X \subseteq v_i \rightarrow \alpha \in v_i)$.

The changes made in the fuzzy case are,
1. Consider [0, 1] instead of {0, 1} i.e. $\mathscr{F} \xrightarrow{v_i} [0,1]$
2. Take subcollection of v_i

Here for further generalization, we are taking a subcollection of all valuations $\{v_i\}$. Instead of writing $\alpha \in v_i$, we can write, $v_i(\alpha)$. Again, we can write $X \subseteq v_i$ as $\forall x(x \in X \rightarrow x \in v_i)$ and $\forall_{i \in I} v_i$, can be used as inf.



So, the formula becomes,

$$\forall_{i \in I} v_i(\forall_{x \in \mathscr{F}} x(x \in X \rightarrow v_i(x)) \rightarrow v_i(\alpha)).$$

i.e.

$$\alpha \equiv \inf_{i \in I}(\inf_{x \in \mathscr{F}}(X(x) \rightarrow v_i(x)) \rightarrow v_i(\alpha)) = gr(X \models \alpha).$$

This is the algebraic expression of the sentence and $gr(X \models \alpha)$ is the semantic graded of consequence of α from X. This expression gives a number in [0, 1]. in $gr(X \models \alpha)$ is a measure of the strength in which α semantically follows from the premise set X in the context when the sentences (wffs) are fuzzy i.e. many valued.

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