Dynamics of Free-Standing Rigid Slender Structures with Eccentricities: A Step Forward in Aseismic Design

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Abstract

Motivated by intriguing response of free-standing slender structures, the study begins with an in-depth examination of their free-vibration characteristics particularly for those with eccentricities. Overturning and rocking spectra are then constructed and interpreted for both positive and negative eccentricities. These spectra offer valuable insights into various phenomena, including overturning modes, the implications of shaking direction, and the interplay between size and slenderness. The research further develops and presents a straightforward framework, introducing a novel *stability coefficient spectrum* and a *rocking spectrum* in a dimensionless and orientationless format. Despite the existence of bifurcation and fractal overturning boundaries in the long-term response, the simplified spectra originally developed for harmonic base excitation prove effective in the context of seismic safety to serve for both force-based and displacement-based design, at least in the near-field. Consequently, the proposed methodology appears sound for the design of free-standing or rocking-isolated slender systems, especially in light of the deficiencies in current design practices.

1.0 Introduction

Between the 5th and 8th centuries, several earthquakes have led to the toppling of ancient statues and monuments. On the other hand, remarkable seismic stability has been in evidence by many slender structures built between 565 B.C. and 141 A.D. (Thomas et al., 1963; Espinosa et al., 1977; Makris and Kampas, 2016). Similar contrasting observations have been made for various slender systems including towers, elevated water tanks, radioactive shields, and household or hospital equipment during numerous seismic events, for example, the 1906 San Francisco earthquake, the 1952 Arvin-Tehachapi earthquake, the 1960 Chile earthquake, and the 1971 San Fernando earthquake. Also, art objects were overturned during the 1989 Loma Prieta earthquake in California (Nigbor et al., 1994), whereas 'some buildings and elevated highways' tilted/collapsed to one side as 'rigid blocks' after the 1985 Mexico City and the 1995 Kobe earthquakes, (Plaut et al., 1996). Therefore, it is essential to delve into the intricate dynamics that underpin the remarkable resilience of slender structures—ranging from unanchored electrical equipment and nuclear facilities to monuments—that, are often eccentric.

In the area of slender free-standing systems, a number of researchers, following the seminal work of Housner (1963), have studied behaviour of free-standing symmetric rigid blocks. The highly nonlinear governing

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equation of motion is known to change with the sense of angle of rotation, with a distinct reduction (discontinuity) in tilting velocity after each impact owing to energy loss. Building upon Housner's insights, numerous studies have rigorously examined the response of free-standing rigid blocks to various excitation types, including (i) half-sine pulses (Housner, 1963; Makris and Roussos, 1998; Makris and Roussos, 2000), (ii) one-sine pulses (Makris and Roussos, 1998; Makris and Roussos, 2000; Zhang and Makris, 2001), (iii) mathematical wavelets (Zhang and Makris, 2001; Vassiliou and Makris, 2012; Makris and Vassiliou, 2012; Dimitrakopoulos and DeJong, 2012a; Makris, 2014a) and (iv) real ground motions (e.g., Yim et al., 1980; Makris and Roussos, 1998; Makris and Roussos, 2000; Zhang and Makris, 2001; Makris and Kampas, 2016). These investigations have shown that overturning might occur in (a) free vibration period after the expiration of the excitation and not essentially at the instant pulse expires, (b) more than one mode, characterized by the number of impacts (Makris and Roussos, 2000; Zhang and Makris, 2001; Vassiliou and Makris, 2012; Makris and Vassiliou, 2012; Dimitrakopoulos and DeJong, 2012b; Makris, 2014b).

The inherent nonlinearity of the problem has prompted researchers to study the long-term behaviour (e.g., Spanos and Koh, 1984; Hogan,1989; Tso and Wong, 1989a, 1989b; Hogan, 1990; Yim and Lim,1991a; Hogan, 1994; Ageno and Sinopoli, 2005; Jeong and Yang, 2012) and revealed the sensitivity of response to initial conditions, excitation characteristics, and block geometry, especially in three-dimension (Pradhan et al., 2022). These studies under harmonic base excitation have (a) identified the safe and unsafe zones per stability analysis (Spanos and Koh, 1984), (b) proposed the concept of stability boundaries (Hogan, 1989; Hogan, 1990), (c) introduced the condition of heteroclinic bifurcation (Hogan, 1989), and (d) found the appearance of the chaos and fractal basin boundaries (Bruhn and Koch, 1991; Yim and Lim, 1991a; Yim and Lim, 1991b; Ageno and Sinopoli, 2005). Researchers (Iyengar and Manohar, 1991; Lin and Yim, 1996a; Lin and Yim, 1996b) have also confirmed these findings for deterministic and stochastic excitation.

Although the studies on the dynamics of rigid blocks are numerous, a few have focused on the asymmetric systems, which are crucial for practical applications (Pradhan and Roy, 2023; 2024). Physically, the mass moment of inertia of a block about the point of impact differs depending on whether the block impacts at one corner or the other, and this can lead to a significant change in response. Limited works on eccentric systems have included (a) those representing statue-pedestal and hospital equipment (e.g., Wittich et al., 2016; Saifullah and Wittich, 2021), (b) earthquake-like base excitations (e.g., Shi et al., 1996; Contento and Di Egidio, 2009; Di Egidio and Contento, 2009; Di Egidio and Contento, 2010) and seismic fragility constructions for rigid blocks and natural rock bodies (e.g., Purvance 2005; Purvance et al., 2008a; Purvance et al., 2008b; Veeraraghavan, 2015). The distinction in behaviour between symmetric and asymmetric blocks is apparent for both short and long-term responses, even for small eccentricities (e.g., Plaut et al., 1996; Zulli et al., 2012). These have also been confirmed by experimental (Wittich and Hutchinson, 2015; Wittich and Hutchinson, 2017; Arredondo et al., 2019) and finite-element based investigations (e.g., Al Abadi et al., 2019).

Despite continued research studies, design guidelines for free-standing rigid blocks (even for symmetric ones) are yet premature. Based on the 'superficial similarities', Priestley et al. (1978) modelled a rocking block as a single-degree-of-freedom (SDoF) oscillator with constant damping, whose period depends on the amplitude of rocking and this serves as the basis of the existing practice (ASCE 43-05, FEMA P-58-1). Subsequent works highlighting the fundamental flaws in this approach recommended to abandon the codified procedure (Makris and Konstantinidis, 2003), and introduced the concept of rocking spectrum. In the elaborate work of Dar et al. (2016), various shortcomings of the design procedure in ASCE 43-05 are further elucidated, and recommended for rigourous nonlinear dynamic analysis. Despite these authoritative works, the codified guidelines continue to lean on the 'flawed' methodology.

Against this backdrop, the present undertaking begins to develop an in-depth understanding on the dynamics of free-standing rigid systems with eccentricities, through comprehensive case studies under simple trigonometric pulses. Recognizing that a block may overturn with multiple impacts during real seismic events, the research explores the long-term response to harmonic pulses. The study ultimately introduces a dual-framework, combining a newly proposed *stability coefficient spectrum* with a novel *rocking spectrum*, which may respectively be used to estimate intensity of seismic shaking for an astute choice of the factor of safety and the corresponding rotation demand.

2.0 Research Significance

The inherent seismic stability of free-standing columns and blocks, as demonstrated in the constructions from 'more than 2.5 millennia ago by the builders of archaic and classical temples', motivates the development of an appropriate design framework and the adoption of such mechanism to enhance seismic resilience. By systematically addressing the complex dynamics of eccentric systems—often overlooked in existing studies—this research achieves new insights into the behaviour of these structures under seismic excitation. It introduces a novel dual-framework combining a *stability coefficient spectrum* and a *rocking spectrum*, offering a more accurate approach to estimating seismic shaking intensity and corresponding rotation demands. This methodology can serve the requirements of both force-based and displacement-based design strategies, ultimately contributing to more resilient infrastructure. Therefore, this investigation, by advancing the design of free-standing slender structures, has the potential to significantly enhance seismic protection of infrastructure, especially monuments, bridges supported by rocking piers (Makris, 2014a; Makris and Vassiliou, 2014).

3.0 General Specifications of Structures Blocks in 2D

The dynamics of free-standing rigid blocks with eccentricity, as shown in Figure 1a, involves several key phenomena. For a base excitation acting rightward, a minimum amplitude of base excitation ($\ddot{u}_{gx}(\mathrm{t}) \geq g \tan \alpha_1$; g is the gravitational acceleration) is required to uplift. Once rocking commences, for a strong excitation shown, the block initially rotates counter-clockwise (i.e., taken $\theta(t) < 0$) and may overturn without impact about the pivot point *B*. If the excitation is not strong enough to cause overturning, the block rotates in the opposite (clockwise) sense, with the first impact occurring at the pivot point B'. This rocking motion continues until the block overturns or returns to its initial state following free vibration. The governing equations of motion, which describe this dynamic behaviour, are depicted below, assuming sufficient friction to prevent sliding (e.g., Yim et al., 1980; Makris and Roussos, 2000; Kounadis et al., 2013; Veeraraghavan, 2015) and allowing for impacts at pivots *B* and B'. In fact, it has been shown (Di Egidio and Contento, 2009) that there exists a limiting friction primarily dependent on the slenderness beyond which sliding may eventually be prevented for seismic excitation.

3.1 Equations of motion

Consider a rigid block with an eccentricity that is supported by a horizontal foundation. Centre of mass, i.e., CM (denoted as G_s) of the block, situated at a vertical distance of *h* from the base or foundation, does not align with the geometric centre (denoted as *G*). The width of the block is $2b = b_1 + b_2$, where b_1 and b_2 are the semi-bases (as shown in Figure 1a).

For the block on horizontal base, the equation of motion corresponding to counter-clockwise rotation ($\theta(t) < 0$) may be expressed as follows:

$$\left(I_{CM} + mR_1{}^2\right)\ddot{\theta}(t) = -mgR_1 \sin\left(-\alpha_1 - \theta(t)\right) - \frac{\ddot{u}_{gx}(t)}{g} mgR_1 \cos\left(-\alpha_1 - \theta(t)\right) \text{...(1a)}$$

On the other hand, for the clockwise rotation ($\theta(t) > 0$), the equation of motion for the rocking block takes the following form:

$$\left(I_{CM} + mR_2{}^2\right)\ddot{\theta}(t) = -mgR_2 \sin\left(\alpha_2 - \theta(t)\right) - \frac{\ddot{u}_{gx}(t)}{g} mgR_2 \cos\left(\alpha_2 - \theta(t)\right) \text{...(1b)}$$

in which m is the block mass, the semi diagonals $R_1(=\sqrt{{b_1}^2+h^2})$ and $R_2(=\sqrt{{b_2}^2+h^2})$ represent size; $\alpha_1 = \tan^{-1}(\frac{b_1}{h})$ and $\alpha_2 = \tan^{-1}(\frac{b_2}{h})$ are the measures of the slenderness. I_{CM} is the mass moment of inertia about B and may be expressed as

$$I_{CM} = m[\tfrac{1}{3}(b^2+h^2)+(b-b_1)^2] \quad ...(1c)$$

Equations (1a) and (1b) can be expressed as follows in compact form (Wittich and Hutchinson, 2015):

$$\left(I_{CM}+mR_i^{\,2}\right)\ddot{\theta}(t) = -mgR_i\sin\left(\alpha_i sign\big(\theta(t)\big)-\theta(t)\right)-\frac{\ddot{u}_{gx}(t)}{g}mgR_i\cos\big(\alpha_i sign(\theta(t))-\theta(t)\big) \quad ...(1)$$

in which the subscript $i = 1$ or 2 respectively indicates block rotation in the negative (about B) and positive (about B') directions, and sign(θ(t)) corresponds to θ(t) > 0 or θ(t) < 0 for 1 or -1, respectively. Assuming a seamless rotation between two pivot points, B and B', without bouncing, the coefficient of restitution, η, using the principle of momentum conservation, can be estimated as follows (Wittich and Hutchinson, 2015).

$$\eta_i = \frac{1}{(I_{CM}+mR_i^{\,2})}[(I_{CM}+mR_j^{\,2})-m(b_1+b_2)R_j\sin\alpha_j] \quad ...(2)$$

where $i = 1,\ j = 2$ represent an impact at point B', and $i = 2, j = 1$ correspond to that at B.

For reference symmetric block (Figure 1b) with base dimension $b = 2b$ $(b_1 = b_2)$ and $\alpha_1 = \alpha_2 = \alpha = b/h$, equations (1) and (2) are reduced as under.

$$\ddot{\theta}(t) = -p^2\left\{\sin\left[\alpha sign\big(\theta(t)\big)-\theta(t)\right]+\frac{\ddot{u}_g(t)}{g}\cos\left[\alpha sign\big(\theta(t)\big)-\theta(t)\right]\right\} \ldots (3a)$$

$$\eta = (1-1.5sin^2\alpha) \ldots (3b)$$

in which frequency parameter $p = \sqrt{3g/4R}$ where R: size, and α: slenderness of the block. We have solved the equation of motion independent by using the ‘event function’ (Vassiliou and Makris, 2011; Makris and Vassiliou, 2012) (this function detects events such as impact while solving equations, and updates the post-impact velocity by multiplying the coefficient of restitution with the pre-impact velocity) within the Ode45 solver (Matlab V.9.5, 2018).

4.0 Response of In-plane Mass-eccentric Structures

To examine the dynamics of free-standing rigid blocks, *prima facie*, a symmetric block of dimensions 0.5m × 2.5m is chosen as reference, and twelve models with in-plane eccentricities are created. These are shown in Figure 2 in the schematic form for e = ±0.001, ±0.002, ±0.005, ±0.025, ±0.05, and ±0.1 metre. These systems with negative eccentricities are respectively marked as S_1, S_2, S_3, S_4, S_5, and S_6, whereas the corresponding positive counterparts are denoted S_1', S_2', S_3', S_4', S_5', and S_6', respectively. For G_s located to the right of G, the eccentricity is treated positive, when the excitation initially acts to the right.

4.1 Basic observations

For preliminary understanding, response histories are studied by varying amplitudes (a_p) of a one-sine pulse of constant frequency, $\omega_p = 6$ rad/s. Figure 3 shows the response histories for each eccentric model, corresponding to the excitation amplitudes of 2.60 m/s^2, 3.50 m/s^2, 4.25 m/s^2, 4.60 m/s^2, 6.20 m/s^2 and 8.50 m/s^2, respectively (Figure 3a to Figure 3f). A close scrutiny to the response histories may be enlightening to appreciate the implications of eccentricity. For instance, a_p = 2.60 m/s^2, (Figure 3a) cannot initiate uplift or rocking in S_1', whereas the companion system with negative eccentricity, i.e., S_1 overturns *with two impacts* (Mode 2, marked as M_2). Interestingly, for the same a_p, S_2 overturns with two impacts (M_2), whereas S_2' survives. For the same intensity S_3' rocks; however, S_3 overturns following *one impact* (Mode 1, marked as M_1). It is noteworthy that the other selected systems, including those with smaller eccentricities, overturn with one impact (M_1).

The effect of a_p may be further appreciated. For a_p = 3.50 m/s^2, S_1 and S_1' overturn without impact (Mode 0, denoted as M_0) and with one impact (M_1), respectively, whereas S_2' and S_3' overturn with *one impact* (M_1), in contrast to the responses for a_p = 2.60 m/s^2 depicted above. For small eccentricities, the response scenario, for selected other values of amplitudes, are similar to those for a_p = 2.60 m/s^2. With increase of amplitude, overturning mode may alter and no systematic trend appears. Strikingly, S_2 survives for a_p = 4.60 m/s^2, while the same overturns at a lower intensity (for a_p = 3.50 m/s^2 and 4.25 m/s^2). At a very high intensity (a_p = 8.50 m/s^2), all systems are observed to overturn without impact (M_0) (Figure 3f).

The limited studies reveal that, for one-sine pulse, the response of eccentric systems is sensitive to both characteristics of eccentricity and the direction of excitation. The overturning phenomenon may be abruptly influenced by the governing modes of overturning, denoted as M_0, M_1 and M_2, which correspond to the number of impacts (zero, one or two) before overturning occurs. Notably, planar blocks with symmetry typically overturn with either one impact or no impact under a one-sine pulse (e.g., Makris and Kampas, 2016). The results indicate that the dynamics of eccentric blocks can differ significantly from their symmetric counterparts, suggesting that the findings from simplified symmetric systems may not be directly applicable to many real-world free-standing systems.

4.2 Observations through free vibration characteristics

It has been previously concluded that planar symmetric blocks subjected to one or half-sine pulse (Makris and Roussos, 2000; Zhang and Makris, 2001; Konstantinidis and Makris, 2007, Dimitrakopoulos and Dejong, 2012a; Dimitrakopoulos and Dejong, 2012b; Dejong and Dimitrakopoulos, 2014) generally overturn in the free vibration regime upon the expiration of the pulse. Hence, it is of interest to examine the same for eccentric blocks. To this end, the response of a free-standing rigid block ($2b$ = 0.5m, $2h$ = 2.5m; e = - 0.025m) to a one-sine pulse with ω_p = 7.6 rad/s, as shown in Figure 4, is studied. Representative response histories are plotted for different excitation amplitudes corresponding to 2.70 m/s^2, 3.10 m/s^2, and 6.89 m/s^2, respectively. The results reveal that the in-plane eccentric block may overturn, either with (two/one) or without impacts, in the free-vibration regime after the completion of the forced vibration. This motivates to systematically study the overturning condition of eccentric rigid blocks using free-vibration characteristics.

Against this backdrop, free-vibration is studied by varying the initial conditions $(\theta_0, \dot{\theta}_0)$ for selected systems, viz., S_1, S_2, S_3, and S_4, and the domains of rocking and overturning in different modes are identified. The domains are graphically presented in Figure 5 in terms of nondimensional initial conditions, i.e., $(\theta_0/\alpha, \dot{\theta}_0/p)$, in which α (= 0.1978 rad) and p (= 2.4024 rad/s) correspond to the related symmetric system. The results show that, for negative θ_0, two overturning modes, viz., M_0, and M_2 appear for large eccentric systems, while M_1 appears with M_2 as the eccentricity reduces. This zone of M_2 is eventually eliminated with further reduction of eccentricity. On the other hand, for positive θ_0, only M_0 and M_1 regulate the overturning phenomena. Hence, the rocking zone and overturning characteristics depend on the initial conditions and the magnitudes of eccentricity. The similarity of the trends to those in the preceding section, suggests that the free-vibration analysis with the initial state variables, corresponding to those at the end of the forced vibration phase, can provide important insight into the stability of block.

With this understanding on the dynamics of eccentric systems, the stability and performance of the free-standing structures subjected to coherent pulse are investigated in the following section.

4.3 Observations through forced vibration

The preceding results suggest that the response of the blocks is sensitive to various factors, including the eccentricities, and direction of excitation and its characteristics etc. The highly nonlinear behaviour is therefore further examined through overturning and rocking spectra, which are widely used for symmetric planar systems. These are then oriented to design goal.

4.3.1 *Overturning spectra and implications*

In this part, the responses to a single sine pulse are computed for systems S_1 through S_6 (and S_1' through S_6'). The response is plotted in Figure 6, with the excitation parameters normalized by the characteristic parameters of the reference symmetric block, viz., α and *p*. These describe the stability of the blocks in terms of its intensity and frequency of the forcing function. In the resulting plots, known as *overturning spectra*, the responses of the companion systems with equal eccentricity but in opposite sense (with respect to the direction of excitation), for example, S_1 and S_1', are also superimposed for comparison.

Figure 6a shows that for S_1', overturning occurs either with one impact (M_1) or no impact (M_0), while the same for S_1 might occur following two impacts (M_2) and no impact (M_0). The uplift strength of the blocks S_1 and S_1' differs because of the variations in α_1. Notably, the block corresponding to S_1 demonstrates greater stability than S_1' at higher excitation frequencies (large columns). A look at the responses to the systems with lesser eccentricities, viz., S_2 and S_2' reveals all three overturning modes (M_0, M_1, M_2) for S_2, whereas, in contrast, S_2' shows two possible modes (M_0, M_1) like S_1'. An interesting observation is that the one-impact region (M_1) appears within the two-impact region (M_2) for S_2, indicating a shift of overturning mode within the range of amplitude of overturning itself. A close scrutiny to the results of the selected cases uncovers the following.

- The systems with large negative eccentricity may overturn in the M_0 and M_2 modes. However, M_1 appears to regulate partly within the M_2 region for systems with moderate negative eccentricity. As the eccentricity continues to decrease, the area covered by M_1 gradually expands, effectively overlapping and eventually suppressing the M_2 region.
- The response of systems with positive eccentricity is, however, relatively straightforward and follows a pattern as that of a symmetric system even though the overturning occurs at a lower amplitude.

- The observed jump to a higher acceleration for overturning suggests a transition to a different overturning mode, and may be attributed to the bifurcation phenomenon in the nonlinear system. For the base excitation in the form of a one-sine pulse, M_2 appears only in systems with negative eccentricity.

Physically, when a block is uplifted, its rotational inertia has a feeble effect for long-duration pulses because the block rotates slowly. Conversely, rotational inertia can play a significant role during short-duration pulses. Since rotational inertia is proportional to the square of the size, stability may be enhanced for high-frequency pulses. As such, the results indicate, for blocks with eccentricity, that 'the increase in the column size can not only offset the anticipated decrease in the column stability due to the increase of slenderness, but on some occasions, it may appreciably increase its stability'. This was indeed confirmed by Makris and Kampas (2016) for blocks with symmetry. For a simpler realization, assuming that the limiting condition for overturning is attained as the centre of mass is just vertically above the point of impact, by limit equilibrium, the displacement capacity can be expressed as follows.

$$u_{max.} = R_1 \sin\alpha_1 = b_1 \text{ overturning about Pivot } B \quad \ldots(4a)$$

$$u_{max}' = R_2 \sin\alpha_2 = b_2 \text{ overturning about Pivot } B' \quad \ldots(4b)$$

These values, in the simplest form, represent that the capacity of the free-standing block is a combination of size, slenderness which in turn, depending on the pivot, changes with eccentricity. It may also be inferred, for a rightward quasi-static excitation (as chosen for overturning spectra in Figure 6), capacity for overturning with zero impact, i.e., about Pivot B is greater for positive eccentricity than that for negative one. Accordingly, for overturning without impact, amplitude of excitation is expected to be greater for positive eccentricity. This is evident from the overturning spectra in Figure 6. These two capacities (as in equations (4a) and (4b)) tend to be identical for $e \sim 0$, resulting in a common value. The free-vibration response characteristics in terms of positive and negative values of θ_0, as discussed already, complement these broad trends.

4.3.2 *Rocking spectra and implications*

The preceding results reveal the complex characteristics related to the stability of a free-standing slender system. From design perspectives, even when a block can sustain a given shaking, it may also be important to regulate the amplitude of the rocking $\theta_{max.}$ (rotation demand). This can be obtained from the rocking spectrum that presents the peak rotation (or angular velocity) as a function of the period of the block for a specified value of slenderness (Makris and Konstantinidis, 2003). As such, rocking spectra, established for symmetric blocks per nonlinear response history analysis in the literature (e.g., Makris and Konstantinidis, 2003; Dimitrakopoulos and DeJong, 2012a), are useful and are viewed as "an additional measure of earthquake intensity" (Makris and Kampas, 2016). Usually, given an excitation, the rocking spectra is presented as a function of the 'period' $T = {}^{2\pi}/_{p}$ for a specified slenderness α.

Taking a_p = 8.00 m/s^2 and ω_p = 12.00 rad/s, the rocking spectra for S_1 to S_5 and S_1' to S_5' are constructed for two selected values of α (= 0.1692 rad and 0.2308 rad) of reference symmetric systems. These are presented in Figure 7. In general, the rotations of rocking blocks asymptotically decrease with the increase of T that corresponds to increasing size of the block. The results also show that at $2\pi/p$ = 3.2, all blocks overturn for α = 0.1692 rad; however, those other than S_1', S_2' and S_3' remain stable when α = 0.2308 rad, indicating that the stability is dependent on both size, slenderness and sense of eccentricities. These are indeed implicit in equation (4).

It may be noted that the format for the rocking spectrum shown in Figure 7, as a function of T for a specified value of slenderness α, is standard and commonly used. To improve upon, the rocking spectra for a selected excitation are reconstructed in a more compact form in Figure 8. This includes blocks with varying slenderness and size, with rotation magnitude represented by different colours. The zones of overturning are also superimposed, providing a consolidated representation of the entire phenomenon of free-standing blocks. The results suggest that, the 'competition' between size and slenderness to regulate overturning is predominant for blocks with relatively small size. However, as block size increases, rocking overrides overturning, yet both factors continue to influence the maximum rotation. A comparison of the spectra for different eccentricities reveals the occurrence of various overturning modes, similar to those observed in section 4.3.1.

From the design and performance assessment perspectives, the acceleration required to prevent overturning, along with the estimate of maximum rotation, are crucial. The paper henceforth aims to achieve this goal.

4.3.2.1 Rocking spectra revisited

To arrive at a more generalized and efficient format for rocking spectra for a simple cycloidal pulse with amplitude a_p and period $T_p = \frac{2\pi}{\omega_p}$ that characterize the spatial and temporal scales of the pulse, respectively, the maximum rotation $\theta_{\text{max.}}$ (taking absolute value only) of a free-standing rigid block with in-plane eccentricity can be mathematically expressed as follows.

$$\theta_{max.} = f(tan\alpha, p, a_p, 2\pi/T_p, g, e) \qquad \ldots(5a)$$

According to Buckingham's Π-theorem, 'if an equation involving u_m variables is dimensionally homogeneous, it can be reduced to a relationship among ($u_m - u_n$) independent dimensionless Π products where u_n is the minimum number of reference dimensions required to describe the physical variables' (Barenblatt, 1996). In the present context, taking g and T_p as repeating variables (the product of these variables is not dimensionless), equation (5a) may be recast in the dimensionless format as below.

$$\Pi'_\theta = \phi(\,\Pi'_\alpha, \Pi_p, \Pi'_e, \Pi'_g) \qquad \ldots(5b)$$

in which $\Pi'_\theta = \theta_{max.}$; $\Pi'_\alpha = tan\alpha$; $\Pi_p = \frac{2\pi}{pT_p} = \frac{\omega_p}{p}$; $\Pi'_e = \frac{e}{gT_p^2}$; $\Pi'_g = \frac{a_p}{g}$. For a more appropriate set of variables, the principles of orientational analysis (Siano, 1985; Araneda, 1996; Dimitrakopoulos and DeJong, 2012a, Roy and Santra, 2024) is further adopted. It may be noted that the orientational analysis cannot, in principle, result in a definite combination; however, can lead to the possible forms conforming to the natural physics of orientational homogeneity. Accordingly, we choose to reorganize equation (5b) as follows.

$$\Pi_\theta = \varphi\left(\Pi_p, \Pi_e, \Pi_g\right) \qquad \ldots(5c)$$

where $\Pi_\theta = \frac{\theta_{max.}}{tan\alpha}$; $\Pi_p = \frac{\omega_p}{p}$; $\Pi_e = \frac{e}{gtan\alpha T_p^2}$; $\Pi_g = \frac{a_p}{gtan\alpha}$. Taking l_x, l_y, l_z as the unit orientations along the x, y, z and l_0 orinetationless quantity, the orientational homogeneity for (5c) can easily be verified. In fact, for the chosen combination, each group in itself is orientationless as shown below.

$$\Pi_\theta = \frac{\theta_{max.}}{tan\alpha} \circeq \frac{l_y}{l_y} = l_0;\ \Pi_p = \frac{\omega_p}{p} \circeq \frac{l_0}{l_0} \circeq l_0;\ \Pi_e = \frac{e}{gtan\alpha T_p^2} \circeq \frac{l_x}{l_z l_y} \circeq l_0,\ \Pi_g = \frac{a_p}{gtan\alpha} \circeq \frac{l_x}{l_z l_y} \circeq l_0$$

This follows that the rocking spectrum established in terms of the variables in (5c) (i) is independent of the absolute magnitude of each characteristics variable but in groups, (ii) reduces the number of influential factors, and (iii) inherently satisfies natural and fundamental physics. Similar development for symmetric systems is available elsewhere (Dimitrakopoulos and Paraskeva, 2015). Using these combinations of dimensionless and orientationless group of variables, rocking spectra are constructed in Figure 9 which display, for constant values of Π_e and Π_g, the variation of Π_θ as a function of Π_p. Π_e = 0 represents the symmetric system, whereas the values of the nondimensional eccentricities are taken as Π_e = ± 0.034 in positive and negative sense, respectively in the sample form. The dispersion among the curves (plotted on the left) appears noticeable even for constant values of Π_e and Π_g. To gain insight, a companion plot, obtained by setting η to a constant (= 0.94), is shown to the right of each curve. These reveal appreciable reduction in dispersion among the variation curves and suggest that the curves approximately collapse to two bands respectively for $tan\alpha \leq 0.26$ (representing many historical monuments as reported in Makris and Kampas, 2016) and above, each for symmetry, positive and negative eccentricities (shown in Figure 9b). Hence, the rocking surfaces as a function of the variables in (5c) should be constructed, in principle, for specific values of $tan\alpha$.

Hence, rocking surfaces in Figure 10a ($tan\alpha$ = 0.172) and Figure 10b ($tan\alpha$ = 0.234) may be attractive for practical assessment as these apply to a broad set of structures subjected to a number of characteristic groups rather than each variable separately. Hence, for a constant Π_e $(= \pm 0.034)$, we construct rocking spectra in the form of surfaces which present the variation of Π_θ as a function of Π_p and Π_g for two different values of $tan\alpha$. Given that the direction of excitation is unknown in advance, maximum values of Π_θ, out of positive and negative Π_e, are plotted. The cross-section of the surfaces for Π_g = 2.000 are also shown in two-dimensional format for convenience (in Figure 10c). These surfaces, corresponding to appropriate Π_e and $tan\alpha$ may be utilized to estimate the value of $\theta_{max.}$, regardless of the values of other governing parameters. The applicability of these spectra for a real-life problem has been illustrated subsequently.

5.0 Stability Coefficient Spectra: Assessing Margin between Overturning and Uplift

The overturning and rocking spectra reveal that, at least for pulse-like record, the response of the block depend on both a_p and ω_p (or $T_p = 2\pi/\omega_p$). It is important to note that $a_{p,OT}$ (acceleration amplitude for overturning) for an eccentric block when subjected to a pulse with a given period T_p may be mathematically expressed as follows.

$$a_{p_{OT}} = f(tan\alpha, b, g, \omega_p, e) \quad \ldots\ldots(6a)$$

In the form of dimensionless groups, this may be expressed as follows.

$$\Pi'_{a_{p_{OT}}} = \phi(\, \Pi'_\alpha, \Pi'_b,\, \Pi'_e) \quad \ldots\ldots(6b)$$

where $\Pi'_{a_{p_{OT}}} = \frac{a_{p_{OT}}}{g}$, $\Pi'_\alpha = tan\alpha$, $\Pi'_b = \frac{b\omega_p{}^2}{g}$, $\Pi'_e = \frac{e\omega_p{}^2}{g}$. Further, equation (6b) can be organized in dimensionless-orientationless form as

$$\Pi_{a_{p_{OT}}} = F'(\Pi''_b, \Pi''_e) \quad \ldots\ldots(6c)$$

in which $\Pi_{a_{p_{OT}}} = \frac{a_{p_{OT}}}{gtan\alpha}$, $\Pi''_b = \frac{b\omega_p{}^2}{gtan\alpha}$, $\Pi''_e = \frac{e\omega_p{}^2}{gtan\alpha}$. Substituting $b = Rsin\alpha$ for a symmetric reference block and $\omega_p = 2\pi/T_p$, after algebraic manipulation, equation (6c) may be expressed as follows.

$$\Pi_{a_{p_{OT}}} = F(\Pi_{rel.}, \Pi_e) \quad \ldots\ldots(6d)$$

where $\Pi_{rel.} = \frac{gtan\alpha T_p{}^2}{Rsin\alpha}$ and $\Pi_e = \frac{e}{gtan\alpha T_p{}^2}$. Notably, b = $Rsin\alpha$ may be viewed, in the simplest form, as the capacity of a symmetric block (as in equation (4)), whereas $gtan\alpha T_p{}^2$ is the minimum characteristics scale of excitation necessary for uplift of the reference symmetric block. Hence, the eccentric block must uplift for this characteristic scale at least for any one direction of shaking. The intensity of the pulse that the block can sustain post-uplift, i.e., $\Pi_{a_{pOT}} = \frac{a_{pOT}}{gtan\alpha}$ is expressed, for a specified value of Π_e, as a function of $\Pi_{rel.}$ – a relative measure of characteristics scale at uplift with respect to the block capacity. This spectrum expressing the amplitude coefficient of an excitation for safe rocking is named herein as is the *stability coefficient spectrum*.

Overturning spectrum reveals that the block may remain stable at higher amplitudes owing to the shift of the overturning modes; however, such possibilities are less likely for earthquake excitations that contain several frequencies, and hence the minimum amplitude out of all probable modes of overturning is taken to establish this *stability coefficient spectrum*. Further, it has already been observed that the values of $\Pi_{a_{pOT}}$ depend on the sense of eccentricities. Since the direction of excitation is generally unknown in advance, the lower value of $\Pi_{a_{pOT}}$, between those for the positive and negative eccentricities, is considered. It is exciting that the *stability coefficient spectrum* plotted in Figure 11 are found to be reasonably crowded for tanα ≤ 0.26.

Figure 11 shows that $\Pi_{a_{pOT}}$ sharply reduces with increase of $\Pi_{rel.}$. The observed trend with $\Pi_{rel.}$ implies that a pulse capable of initiating rocking may induce overturning, depending on the duration of the pulse. For a column subjected to static axial compression, the load required for the loss of stability decreases as slenderness increases, whereas the system becomes very stable as slenderness decreases. For a free-standing slender block, $\Pi_{rel.}$ may be viewed as analogous to the slenderness of the block relative to excitation. Since a long-duration pulse corresponding to a larger $\Pi_{rel.}$ induces a weak angular acceleration, the rotational inertia is weakly engaged, and as a result, the block is more prone to overturning upon uplift with little margin for stability. On the other hand, the block is likely to remain stable when $\Pi_{rel.}$ is relatively small as the rotational inertia strongly contributes.

The *stability coefficient spectrum* may provide reasonable estimate of the safety margin that a block enjoys between uplifting and overturning. For a given excitation (strong enough to uplift) with duration T_p, the intensity of excitation $a_{p,OT}$ for overturning a block may be estimated for known values of eccentricity. In the context of design, the maximum permissible amplitude ($a_{p,allow.}$) may be obtained for a chosen factor of safety (β) as $a_{p,allow.} = a_{p,OT}/\beta$. It may be noted that, for a pulse with intensity less than $a_{p,OT}$, even though the block will not overturn, may experience objectionable rotation. Thus, to ensure acceptable performance, the corresponding rotation demand may be read from the rocking spectrum and β may accordingly be adjusted. This is illustrated in the next section.

6.0 Design for Free-standing Rocking Blocks: Illustration

The *stability coefficient spectra* proposed in Figure 11 together with the rocking spectra in the dimensionless and orientationless form may be useful for design of slender blocks. This has been illustrated through a simple example in **Appendix I**. For a given block and excitation (strong enough for uplift) having duration T_p, $\Pi_{rel.}$ may be easily calculated and $\Pi_{a_{p_{OT}}}$ can then be readily obtained from the stability coefficient spectrum of relevant Π_e. The intensity of excitation $a_{p,OT}$ for overturning may, hence, be directly read from the *stability coefficient spectra*. In the context of design, the maximum permissible amplitude ($a_{p,allow.}$) may be obtained for a chosen factor of safety (β) as $a_{p,allow.} = a_{p,OT}/\beta$. Clearly, β = 1 represents the curve corresponding to just overturn. It may be noted that, for the pulse with $a_{p,allow.}$ so estimated, even though the block will not overturn, may experience substantial rotation. Thus, to limit maximum rotation to a selected performance level, the corresponding rotation be checked from the associated rocking spectrum. **Appendix I** summarises the procedures for design and performance assessment using the stability coefficient spectra in conjunction with the rocking spectrum in details.

It is apparent that the proposed method can be applied with confidence, especially when the base excitation is harmonic as may often occur in mechanical systems. In reality, a seismic excitation may be more involved; however, kinematic characteristics of ground motions in the near-field often reveal distinguishable acceleration pulse, which may be characterized by a_p and ω_p, respectively representing the pulse acceleration amplitude and its frequency. These parameters, for a given seismic excitation, may be formally extracted using the established mathematical technique that engages wavelet transforms (Vassiliou and Makris, 2011). For a given block and a selected excitation strong enough for uplift, the intrinsic value of $\Pi_{rel.}$ may be suitably calculated and $a_{p,allow.}$ and $\theta_{max.}$ may be similarly estimated. Thus, the proposed design procedure may be useful for seismic design, especially in the near-field for records with coherent pulse. This, however, warrants further scrutiny and is made in the following section.

7.0 Application to Seismic Shaking

The overturning spectra for a one-sine pulse explicitly recognize the existence of different modes of overturning, each associated with distinct behaviour characterized by no impact, one impact, or two impacts. In the context of seismic shaking, the response histories of a block covering both positive and negative eccentricities ($e = \pm\, 0.025$m) are computed. The values of $\alpha = 0.1857$ rad, $p = 1.6566$ rad/s are taken with reference to the historical column in Temple of Aphaia, Aegina (Makris and Kampas, 2016). The response histories are plotted in Figure 12 for an arbitrarily selected real motion scaled to different values — corresponding to rocking and overturning. A look at the histories indicates that the overturning in a seismic event may occur following a few impacts.

To gain insight into the behaviour associated with multiple impacts, the response of a block is studied for successive harmonic excitations continued over a long duration of 150s, taking increments of a_p and ω_p as 0.0186 m/s^2 and 0.0053 rad/s, respectively, as shown in Figure 13. The regions of overturning are segregated by different colours according to the number of impacts before overturning, while the regions for rocking are left white. It is evident that

- the block may overturn in multiple modes (classified herein as $M_{(0)}$, $M_{(1)}$, $M_{(2)}$, $M_{(3\text{-}7)}$, $M_{(8\text{-}14)}$, $M_{(15\text{-}35)}$, and $M_{(>35)}$, in which numbers in the bracket represent numbers of impact), in contrast to the characteristics for a one-sine pulse. Accordingly, several modes of overturning are classified and designated as $M_{(0)}$, $M_{(1)}$, $M_{(2)}$, $M_{(3\text{-}7)}$, $M_{(8\text{-}14)}$, $M_{(15\text{-}35)}$, and $M_{(>35)}$.
- the domain of overturning is sensitive to the direction of eccentricity, and
- the boundaries delineating the overturning zones with different number of impacts are fractal and are extremely sensitive as such.

Furthermore, a careful examination of the bifurcation diagram (Figure 14 and Figure 15) for symmetric and eccentric blocks reveals that multiple solutions are possible with changes in the amplitude parameter of excitation. This, combined with the irregular basin of attraction and the emergence of even and odd subharmonics, underscore the sensitivity to initial conditions (Figure 16 and Figure 17). These findings highlight the need to re-evaluate the performance of the proposed design approach exclusively for seismic shaking; even though the earthquake records typically contain a wide variety of frequencies, last over short duration resulting in a few impacts prior to overturning. Therefore, the proposed design guidelines engaging the minimum overturning acceleration, suitable safety factors, and maximum rotation demands are briefly revisited for real records. For this purpose, fifteen near-fault motions with forward-directivity signature are selected from the NGA-WEST2 database of the Pacific Earthquake Engineering Research (PEER) Centre. The details of these records, as available elsewhere (Roy et al., 2018; Roy et al., 2020), are summarized in Table 1.

Based on earlier literature (e.g., Dimitrakopoulos et al., 2009; Efthymiou and Makris, 2022; Roy et al., 2018; Roy et al., 2020; Acharjya and Roy, 2023), taking the mean period of ground motion (T_m as defined in

Rathje et al., 2004) as substitute for T_p, values of $a_{p,OT}$ are computed by nonlinear dynamic analyses and are marked in the proposed stability coefficient spectra for a block with $tan\alpha$ = 0.172. (Figure 18a). Assuming a factor of safety β = 1.8 by inspection, the variation curve for $a_{p,allow}$ is constructed and overlain therein. This shows that β = 1.8 may be reasonable to estimate $a_{p,allow}$ at least for the selected motions. Further, corresponding to $a_{p,allow}$, values of $\theta_{\text{max.,}}$, covering both positive and negative values of eccentricities, as obtained from nonlinear dynamic analysis ($\theta_{\text{max.,NDA}}$) and the proposed rocking surfaces ($\theta_{\text{max.,RS}}$), are computed and presented in Figure 18b. It appears that the rocking surfaces can estimate the maximum rotation generally on the conservative side (excepting three cases only). A close look to Figure 18a and Figure 18b suggests that considering β = 1.8 may appreciably reduce $a_{p,allow}$, preventing uplift, especially for larger $\Pi_{rel.}$ The corresponding cases estimating $\theta_{\text{max.,RS}} \leq 0.1$ are marked by green diamonds; representing negligible $\theta_{\text{max.,NDA}}$. It is already explained that, in this region, safety margin between uplift and overturning is insignificant and hence, $a_{p,allow}$ is restricted close to just uplift state resulting in small θ_{max}. For the same set of records, a similar set of results is presented in Figure 19a for $tan\alpha$ = 0.234 which, for β = 1.8 ensures safety for ~ 75%. (~ 2.5 may be adequate to be safe in each case). The values of $\theta_{\text{max.RS}}$ also appear reliable for most of these cases (Figure 19b).

Considering the nonlinear nature of the problem and the complex characteristics of an earthquake record, the performance of the simplified spectra corresponding to the harmonic base excitations appears remarkable. It should also be noted that the current methodology only involves T_m of an earthquake, which can be reasonably estimated using established ground motion models (NIST GCR 11-917-15, 2011; Du, 2017). Therefore, the proposed guidelines, founded on a rational understanding of dynamics and fundamental principles of mechanics, are useful for routine design of free-standing and rocking-isolated slender systems, especially in view of the 'flawed' methodology currently in use. However, it may essential to explore suitable β, may be in terms of $\Pi_{rel.}$ and α, using a larger number of systems and real records.

8.0 Summary and Conclusions

This paper makes a comprehensive investigation into the behaviour of free-standing slender structures with eccentricities. After a scrutiny to free-vibration characteristics, the response to harmonic base excitations is studied through overturning and rocking spectra for both positive and negative eccentricities. The comprehensive case studies provide critical insights into key dynamic phenomena, such as overturning modes, sensitivity to the direction of shaking, and the interplay between the size and slenderness of the blocks. A novel *stability coefficient spectrum* and a *rocking spectrum* in the dimensionless and orientationless format are proposed and these have been shown useful for the design of slender free-standing systems to harmonic pulses and seismic base excitation. The investigation leads to the following broad conclusions.

1. A comprehensive analysis of the response of slender blocks with eccentricity to one-sine pulse, corresponding overturning spectra as well as the free-vibration properties reveals the following.
   - The systems with large negative eccentricity may overturn in the M_0 and M_2 modes. However, M_1 appears to regulate partly within the M_2 region for systems with moderate negative eccentricity. As the eccentricity continues to decrease, the area covered by M_1 gradually expands, virtually crushing M_2 region and assumes the characteristics of symmetric systems.
   - The response of systems with positive eccentricity is similar to those of symmetric ones, i.e., overturning might occur either in M_0 or M_1 mode, but expectedly at a lower amplitude.
   - The observed jump to a higher acceleration for overturning suggests a transition to a different overturning mode, and may be attributed to the bifurcation phenomenon of the nonlinear system. In case of the base excitation in the form of a one-sine pulse, the mode M_2 appears only in the systems with negative eccentricity.
   - For one-sine pulse, overturning (if any) occurs in the free-vibration regime and a look at the free-vibrational characteristics for varying state-variables corroborate the same.
2. The dynamics of the eccentric systems is further examined through rocking spectrum in its standard form, which reveals the sensitivity of response to magnitude and direction of eccentricity. For a specified excitation, rocking spectra are reconstructed for blocks with varying slenderness and size, overlain therein the zones of overturning in the compact form. These together elucidate that the ‘competition’ between size and slenderness to regulate overturning is predominant for blocks with relatively small size. However, as block size increases, rocking overrides overturning, yet both factors continue to influence the maximum rotation.
3. The novel contribution of this paper lies in the development of the stability coefficient spectra, which, for any given block, can be readily employed to estimate the overturning acceleration for a pulse of known duration. These values of overturning acceleration can then be appropriately scaled to determine the allowable intensity. Next, the corresponding maximum rotation of the block can be calculated using the innovative rocking spectra introduced in this work. Both the stability coefficient spectra and the rocking spectra are dimensionless and orientationless, ensuring that they conform to the fundamental principles of physics.

4. The inherent nonlinearity of free-standing systems leads to complex behaviour in their long-term response, resulting in bifurcations and the emergence of fractal boundaries of overturning after varying numbers of impacts. Notably, the proposed strategy, based on simple harmonic base excitation, demonstrates remarkable performance when applied to real seismic excitations, at least in the near field. This approach is found effective to estimate both the allowable intensity of shaking and the corresponding rotation, considering the mean period (T_m) as the appropriate time scale. However, the choice of appropriate factor of safety deserves further investigation.

Given that the vertical component is significantly downscaled while influencing rocking systems, the observations and methodology may still be applicable even when the vertical component is strong (Shi et al., 1996; Makris and Zhang, 1999; Makris and Kampas, 2016).

In summary, this paper makes a comprehensive investigation into the response of free-standing slender structures to base excitation per overturning and rocking spectra for both positive and negative eccentricities. The novel stability coefficient spectra and rocking spectra for harmonic base excitation have been shown invaluable for seismic design of free-standing systems. The illustration shown can be taken forward as convenient design approach. Future work should refine the proposed approach, particularly by developing more accurate guidelines for β. This β might be somewhat greater as the basic spectra hinge on simplified harmonic pulse. Additionally, the methodology warrants further examination in the context of three-dimensional bodies subjected to orthogonal components of shaking - a challenge that may be addressed by drawing upon recent studies in three-dimensional framework (Pradhan et al. 2022; Pradhan and Roy, 2023; 2024). Such studies should also investigate the effects of variation of restitution parameter and eccentricities in view of the inherent uncertainties in their estimation process.

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Data availability:

Some or all data, models, or code that support the findings of this study are available from the corresponding author upon reasonable request.

Appendix I: Design of Free-standing Structures

Objectives: The following example illustrates the strategies for design and performance assessment of a free-standing structure using the *stability coefficient spectra* and *rocking spectra* derived herein in dimensionless and orientationless format.

Illustration: A free-standing rigid block (dimension: $2b = 0.813m; 2h = 4.736m;\ e\ =\ 0.017m$) is subjected to a base excitation with pulse period T_p = 0.550s (anticipated). Determine

(a) $a_{p,OT}, i.e.,$ the maximum amplitude of excitation corresponding to overturning, and

(b) the allowable value of amplitude $a_{p,allow}$ for which the system will rock with at least a factor of safety, β = 1.1 against overturning and maximum permissible rotation $\frac{\theta_{allow}}{tan\alpha} \leq 0.330$.

Also evaluate (c) the response of another block (dimension: $2b = 0.491m, 2h = 2.097m;\ e\ =\ 0.015m$) when subjected to a base excitation with $a_p = 4.594\ m/s^2$ and T_p = 0.442s.

Solution:

Reference symmetric system

Characteristics of Block:

Dimension of block: $2b = 0.813m; 2h = 4.736m$

∴ The semi-diagonal length $R = \sqrt{b^2 + h^2} = \sqrt{0.406^2 + 2.368^2} = 2.403m$

and the slenderness $\alpha = \tan^{-1} b/h = 0.170\ rad$

Characteristics of Block relative to Excitation:

Minimum acceleration amplitude for uplift, $a_{p,up} = gtan\alpha = 9.81 \times tan(0.17)\ = 1.684\ m/s^2$

∴ Taking T_p = 0.550s, the parameter $\Pi_{rel.} = \frac{gtan\alpha T_p^2}{Rsin\alpha} = \frac{1.684 \times 0.550^2}{2.403 \times sin(0.17)} = 1.253$

(a) For e = 0.017m, $\Pi_e = \frac{e}{gtan\alpha {T_p}^2} = = \frac{0.017}{1.684 \times 0.55^2} = 0.034$

So, for $\Pi_{rel.}$ = 1.253, Π_e = 0.034; $\Pi_{a_{p_{OT}}} = \frac{a_{p,OT}}{gtan\alpha}$ = 3.918 (refer to Figure 11)

Hence, the block ($2b = 0.813m; 2h = 4.736m\ , e_x\ =\ 0.017$m), will overturn at $a_{p,OT}$ = 3.918 × 9.81× $tan(0.17)$ = 6.598 m/s^2 for the excitation T_p = 0.550s

(b) Considering β = 1.100, $a_{p,allow} = \frac{a_{p,OT}}{\beta}$ = 5.998 m/s^2

Now for the base excitation with $a_p = a_{p,allow}$ = 5.998 m/s^2 and T_p = 0.550s,

$$\frac{\theta_{max.}}{tan\alpha} = 0.733 \text{ (refer to Figure 10a)} > \frac{\theta_{allow.}}{tan\alpha} = 0.330$$

(derived from the nonlinear response history analysis of the actual asymmetric block or may be obtained from the corresponding rocking spectrum).

<u>Iterations</u>

Iteration No. 1:

To reduce maximum rotation, assume β = 1.200, $a_{p,allow} = \frac{a_{p,OT}}{\beta}$ = 5.498 m/s^2

Now for the base excitation with $a_p = a_{p,allow}$ = 5.498 m/s^2 and T_p = 0.550s,

$$\frac{\theta_{max.}}{tan\alpha} = 0.633 \text{ (refer to Figure 10a)} > \frac{\theta_{allow.}}{tan\alpha} = 0.330$$

Iteration No. 2:

To reduce maximum rotation, assume β = 1.960, $a_{p,allow} = \frac{a_{p,OT}}{\beta}$ = 3.366 m/s^2

Now for the base excitation with $a_p = a_{p,allow}$ = 3.366 m/s^2 and T_p = 0.550s,

$$\frac{\theta_{max.}}{tan\alpha} = 0.319 \text{ (refer to Figure 10a or Figure 10c)} < \frac{\theta_{allow.}}{tan\alpha} = 0.330$$

Therefore, the characteristics of the design acceleration are $a_p = 3.366\ m/s^2$, T_p = 0.550 s for a factor of safety of 1.960 (> 1.100) against overturning as well as for a maximum rotation to ~ 0.319 tanα.

(c) For the excitation with T_p = 0.442,

$$\Pi_{rel.} = \frac{gtan\alpha T_p^2}{Rsin\alpha} = \frac{9.81 \times tan(0.23) \times 0.442^2}{1.077 \times sin(0.23)} = 1.828$$

For $\Pi_{rel.}$ = 1.827, Π_e = 0.034; $\Pi_{a_{p_{OT}}} = \frac{a_{p,OT}}{gtan\alpha}$ = 2.655 corresponding to block $2b = 0.491m, 2h = 2.097m$; e = 0.015 m (refer to Figure 11).

Hence, $a_{p,OT} = \Pi_{a_{p_{OT}}} \times gtan\alpha = 2.655 \times 9.81 \times tan(0.23)$ = 6.098 m/s^2

Hence, the block shall uplift and rock with β = $\frac{a_{p,OT}}{a_p}$ =1.327 against overturning for excitation with $a_p = 4.595\ m/s^2$, T_p = 0.442 s.

Also, for a_p = 4.595 m/s^2 and T_p = 0.442 s, $\frac{\theta_{max.}}{tan\alpha}$ = 0.501 (refer to Figure 10b or Figure 10c) (derived from the nonlinear response history analysis of the actual asymmetric block).

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